F. Cajori

AN

ELEMENTARY TREATISE

ON

GEOMETRY,

SIMPLIFIED FOR

BEGINNERS NOT VERSED IN ALGEBRA.

PART I,

CONTAINING

PLANE GEOMETRY,

WITH ITS APPLICATION TO THE SOLUTION OF PROBLEMS.

BY FRANCIS J. GRUND.

New Edition, stereotyped

BOSTON:
CHARLES J. HENDEE,
AND
G. W. PALMER AND COMPANY.

1838.

RECOMMENDATIONS.

From John Farrar, Professor of Mathematics and Natural Philosophy at Harvard University.

Mr. Grund's Elementary Treatise on Geometry contains much useful matter, not generally to be found in English works of this description. There is considerable novelty, also, in the style and arrangement. The subject appears to be developed in a manner well suited to the younger class of learners, and to such an extent, and with such illustrations, as renders it a valuable introduction to the more extended works on Geometry.

JOHN FARRAR.

FEBRUARY 18th, 1830.

From G. B. Emerson, Principal of the English Classical School, Boston.

Mr. Grund's Geometry unites, in an unusual degree, strictness of demonstration with clearness and simplicity. It is thus very well suited to form habits of exact reasoning in young beginners, and to give them favorable impressions of the science. I have adopted it as a text book in my own school.

GEO. B. EMERSON.

FEBRUARY 18th, 1830.

From E. Bailey, Principal of the Young Ladies' High School, Boston.

DEAR SIR—From the specimens of your work on Geometry which I have seen, and especially from the sheets I have used in my school since it went to the press, I have formed a high opinion of its merits. The general plan of the work appears to be very judicious, and you have executed it with great ability. Simplicity has been carefully studied, yet not at the expense of rigid demonstration. In this respect, it seems admirably fitted for the use of common schools. Believing your work cal-

culated and destined to do much good, in a department of science which has been too long neglected, I hope it may soon become generally known.

Very respectfully, yours, &c.

E. BAILEY.

FEBRUARY 17th, 1830.

From F. P. Leverett, Principal of the Latin School, Boston.

DECEMBER 7th, 1830.

DEAR SIR—I have looked with much satisfaction over the sheets of the second edition of your 'First Lessons in Plane Geometry.' It is a more simple and intelligible treatise on Geometry than any other with which I am acquainted, and seems to me well adapted to the understandings of young scholars.

I am, dear sir, respectfully yours,

F. P. LEVERETT.

From William B. Fowle, Principal of the Monitorial School, Boston.

BOSTON, FEBRUARY 17th, 1830.

MR. GRUND—DEAR SIR—I have examined every page of your 'First Lessons in Plane Geometry.' Its reception every where augurs well for the success of your book, which is an extension and practical application of Fraucoeur's. It has fulfilled my wishes, and I shall immediately introduce it into my school.

Yours, very respectfully,

WILLIAM B. FOWLE.

From Walter R. Johnson, Principal of the Philadelphia High School.

PHILADELPHIA, NOV. 27th, 1830.

DEAR SIR—The First Lessons in Plane Geometry, with a perusal of which I have been favored, appears to me eminently calculated to lay the foundation of a clear and comprehensive knowledge of the demonstrative parts of that important science.

As it has obviously been the result of actual experience in *teaching*, it commends itself to the attention of the profession, by the assurance that it is really adapted to the comprehension and attainments of those for whom it was designed. Permit me to express the hope that it may meet its full share of that encouragement which works in this department are beginning to receive in every part of our country.

I remain, dear sir, very respectfully yours,

WALTER R. JOHNSON.

PREFACE.

Popular Education, and the increased study of Mathematics, as the proper foundation of all useful knowledge, seem to call especially for Elementary Treatises on Geometry, as has been evinced in the favorable reception of the first edition of this work within a few months of the date of its publication. A few changes have been made in the present edition, which, it is hoped, will contribute to the usefulness of the work as a book for elementary instruction.

The author acknowledges with pleasure the valuable aid he has received from some of the most experienced and distinguished instructers; and is, in this respect, particularly indebted to the kindness of Messrs. E. Bailey, George B. Emerson, and Miss Elizabeth P. Peabody, of Boston, at whose suggestion several demonstrations have been simplified, in order to adapt the work to the capacity of early beginners.

As regards the use of it in schools and seminaries, the teacher will find sufficient directions in the remarks inserted in the body of the work.

The Problems, of which the third and fourth parts are principally selected from those of Meier Hirsch, form a section by themselves, in order to be more easily referred to.

The teacher may, according to his own judgment, use as many of them at the end of each section, as may be solved by the principles the pupils have become acquainted with.

BOSTON, *September* 30, 1830.

PREFACE TO THE STEREOTYPE EDITION.

THE present stereotype edition differs from the previous ones only in the typographical arrangement, to meet the view of the publishers, whose intention it is to reduce its price, in order to bring it within the reach of common schools throughout the Union.

F. J. G.

BOSTON, *March* 27, 1832.

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GEOMETRY.

INTRODUCTION.

If, without regarding the qualities of bodies, viz: their smoothness, roughness, color, compactness, tenacity, &c., we merely consider *the space which they fill—their extension in space*—they become the special subject of mathematical investigation, and the science which treats of them, is called *Geometry.*

The extensions of bodies are called *dimensions.* Every body has three dimensions, viz: *length, breadth,* and *depth.* Of a wall or a house, for instance, you can form no idea, without conceiving it to extend in length, breadth, and depth; and the same is the case with every other body you can think of.

The limits or confines of bodies are called *surfaces* (superfices), and may be considered independently of the bodies themselves. So you may look at the front of a house, and inquire how long and how high is that house, without regarding its depth; or you may consider the length and breadth of a field, without asking how deep it goes into the ground, &c. In all such cases, you merely consider *two* dimensions. *A surface is, therefore, defined to be an extension in length and breadth without depth.*

The limits or edges of surfaces are called *lines*, and may again be considered independently of the surfaces themselves. You may ask, for instance, how *long* is the front of such a house, without regarding its height; or how far is it from Boston to Roxbury, without inquiring how *broad* is the road. Here, you consider evidently only *one* dimension; and *a line, therefore, is defined to be an extension in length without breadth or depth.*

The beginning and end of lines are called *points* They merely mark the *positions* of lines, and can, therefore, of themselves, have no magnitude. To give an example: when you set out from Boston to Roxbury, you may indicate the place you start from, which you may call the *point* of starting. If this chances to be Marlborough Hotel, you do not ask how *long*, or *broad*, or *deep* that place is; it suffices for you to know the *spot* where you begin your journey. A point is, therefore, defined to be *mere position, without either length or breadth.*

Remark. A point is *represented* on paper or on a board, by a small dot. A line is *drawn* on paper with a pointed lead pencil or pen; and on the board, with a thin mark made with chalk. The extensions of surfaces are *indicated* by lines; and bodies are *represented* on paper or on the board, according to the rules of perspective.

Before we begin the study of Geometry, it is necessary, first, to acquaint ourselves with the meaning of some terms, which are frequently made use of in books treating on that science.

Definitions.

A *line* is called *straight,* when every part of it lies in *the same direction,* thus,

Any line in which no part is *straight,* is called a *curve* line.

A *geometrical plane* is a surface, in which two points being taken at pleasure, the straight line joining them lies entirely in that surface.* A surface in which no part is plane, is called a *curved* surface. Any plane surface, terminated by lines, is called a *geometrical figure.*

The simplest rectilinear figure, terminated by *three* straight lines, is called a *triangle.*

A geometrical figure, terminated by *four* straight lines, is called a *quadrilateral*—by 5, a *pentagon*—by 6, a *hexagon*—by 7, a *heptagon*—by 8, an *octagon,* &c.

Any geometrical figure, terminated by more than three straight lines, is (by some authors) called a *polygon.*†

When two straight lines meet, they form an *angle;* the point at which they meet is called the *vertex,* and the lines themselves are called the *legs* of the angle. When a straight line meets another, so as to make the two adjacent angles equal, the angles are called *right* an-

* The teacher can give an illustration of this definition, by taking anywhere on a piece of pasteboard, two points and joining them by a piece of stiff wire. Then, by bending the board, the wire, which represents the line, will be off the board, and you have a curved surface; and by stretching the board, so as to make the wire fall upon it, you have a plane.

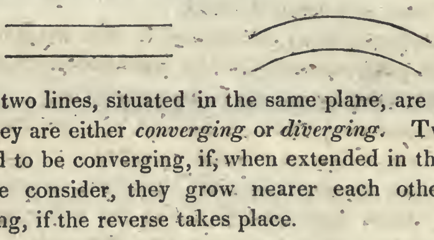
† Legendre calls all geometrical figures polygons.

gles, and the lines are said to be *perpendicular* to each other.

Any angle smaller than a right angle is called *acute*,

and when greater than a right angle, an *obtuse* angle.*

Two lines which, lying in the same plane, and however far extended in both directions, never meet, are said to be parallel to each other.



When two lines, situated in the same plane, are not parallel, they are either *converging* or *diverging*. Two lines are said to be converging, if, when extended in the direction we consider, they grow nearer each other; and diverging, if the reverse takes place.

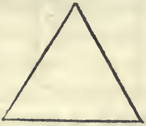
Converging.

Diverging.

---

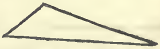
* Angles are measured by arcs of circles, described with any radius between their legs. Here the teacher may state, that the circle is divided into 360 equal parts, called degrees; each degree, again, into 60 equal parts, called minutes; a minute, again, subdivided into 60 equal parts, called seconds, &c.; and that the magnitude of an angle can thus be expressed in degrees, minutes, seconds, &c. of an arc of a circle, contained between its legs.

A triangle is called *equilateral*, when all its sides are equal.

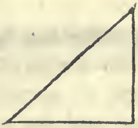


A triangle is called *isosceles*, when *two* of its sides only are equal.

A triangle is called *scalene*, when none of its sides are equal.



A triangle is also called *right-angled*, when it contains a *right angle;*



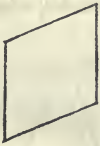
and *oblique-angled*, when it contains no right angle.

A *parallelogram* is a quadrilateral whose opposite sides are parallel.

A *rectangle*, or *oblong*, is a right-angled parallelogram

A *square* is a rectangle whose sides are all epual.

A *rhombus* or *lozenge*, is a *parallelogram* whose sides are all equal.

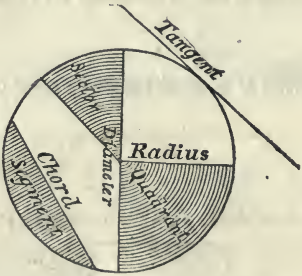


A *trapezoid* is a quadrilateral in which two sides only are parallel.

A straight line joining two vertices, which are not on the same side of a geometrical figure, is called a *diagonal.*

The side which is opposite to the right angle, in a right-angled triangle, is called the *hypothenuse.*

A *circle* is a surface terminated on all sides by a curve line returning into itself, all points of which are at an equal distance from one and the same point, called *the centre.*



The curve line itself is called the *circumference.* Any

part of it is called an *arc*. A straight line, drawn from the centre of a circle to any point of the circumference, is called a *radius*. A straight line, drawn from one point of the circumference to the other, passing through the centre, is called a *diameter*. A straight line, joining any two points of the circumference, without passing through the centre, is called a *chord*.

The plane surface included within an arc of a circle and the chord on which it stands is called a *segment.*

The arc of a circle which stands on a diameter is called a *semi-circumference*. The plane surface included within a semi-circumference and a diameter is called a *semi-circle*.

The plane surface included within two radii and an arc of a circle is called a *sector*. (See the figure, page 14.) If the two radii are perpendicular to each other, the sector is called a *quadrant*.

A straight line, which, drawn without the circle, and however far extended in both directions, meets the circumference only in one point, is called a *tangent*.

QUESTIONS ON DEFINITIONS.

What is that science called, which treats of the extensions of bodies, considered separately from all their other qualities?

What are the extensions of bodies called?

What are the limits or confines of bodies called?

How do you *define* a surface?

What are the limits of surfaces called?

How do you *define* a line?

What are the beginning and end of lines called?

How do you *define* a point?

How is a geometrical point *represented?*

How is a line represented? How a surface?

How do you define a *straight* line?

What do you call a line in which no part is straight?

What is that surface called, in which, when two points are taken at pleasure, the straight line joining them lies entirely in it?

What do you call a surface in which no part is plane?

What is a plane surface called when terminated by lines?

By how many straight lines is the simplest rectilinear figure terminated?

What do you call it?

What do you call a geometrical figure terminated by four straight lines?

What, if terminated by five straight lines?

What, if by six? By seven? By eight?

What are all geometrical figures terminated by more than three straight lines called?

When two straight lines meet, what do they form?

What is the point where the lines meet called?

What do you call the lines which form the angle?

If one straight line meets another, so as to make the two adjacent angles equal, what do you call these angles?

What are the lines themselves said to be?

What is an angle which is smaller than a right angle called?

What an angle larger than a right angle?

What do you call two lines, which, situated in the same plane, and however far extended both wavs never meet?

When are two lines said to be converging? When, diverging?

When a triangle has all its sides equal, what is it called?

When two of its sides only are equal, what?

When none of its sides are equal, what?

What is a triangle called, when it contains a right angle?

What, if it does not contain one?

What is a quadrilateral, whose opposite sides are parallel, called?

What is a right-angled parallelogram called?

What is an equilateral rectangle called?

What, an equilateral *parallelogram?*

What, a quadrilateral in which two sides only are parallel?

How is a circle terminated?

What is the line called which terminates a circle?

What is any part of the circumference called?

What, a straight line, drawn from the centre, to any point in the circumference?

What, a straight line joining two points of the circumference, and passing through the centre?

What, a straight line joining two points of the circumference, without passing through the centre?

What is the plane surface, included within an arc and the chord which joins its two extremities, called?

What is that part of the circumference called, which is cut off by the diameter?

What, the plane surface within a semi-circumference and a diameter?

What, the surface within an arc of a circle and the two radii drawn to its extremities?

What is the sector called, if the two radii are perpendicular to each other?

What is the name of a straight line, drawn without the circle, which, extended both ways ever so far, touches the circumference only in one point.

NOTATION AND SIGNIFICATIONS.

For the sake of shortening expressions, and thereby to facilitate language, mathematicians have agreed to adopt the following signs:

$=$ stands for equal; *e. g.*, the line $AB = CD$ means, that the line AB is equal to the line CD.

$+$ stands for *plus* or *more; e. g.*, the lines $AB + CD$ means, that the length of the line CD is to be added to the line AB.

$-$ stands for *minus* or *less; e.g.*, line $AB - CD$ means, that the length of the line CD is to be taken away from the line AB.

$\times$ is the sign of multiplication.

: is the sign of division.

$<$ stands for *less than; e.g.*, the line $AB < CD$ means, that the line AB is shorter than the line CD.

$>$ stands for *greater than; e. g.*, the line $AB > CD$ means, that the line AB is longer than the line CD.

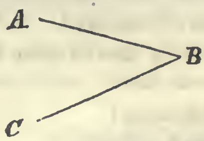
A point is denoted by a single letter of the alphabet chosen at pleasure; *e. g.*,

the point B.

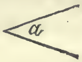
A line is represented by two letters placed at the beginning and end of it; *e. g.*,

the line AB.

An angle is commonly denoted by three letters, the one that stands at the vertex always placed in the middle;

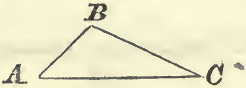


the angle ABC or CBA. It is sometimes also represented by a single letter placed within the angle; *e. g.*,



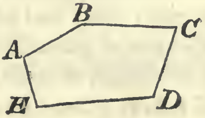
the angle *a*.

A triangle is denoted by three letters placed at the three vertices; *e. g.*,



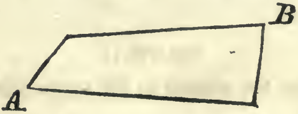
the triangle ABC.

A polygon is denoted by as many letters as there are vertices; *e. g.*,



the pentagon ABCDE.

A quadrilateral is sometimes denoted only by two letters, placed at the opposite vertices; *e. g.*,



the quadrilateral AB.

QUESTIONS ON NOTATION AND SIGNIFICATIONS.

What is the sign of equality?
What sign stands for plus or more?
What for minus or less?
What for multiplication?
What for division?
What for *less than?*
What for *more than?*
How is a point denoted?
How a line?
How an angle?
How a triangle?
How a quadrilateral?
How any polygon?

Axioms.

There are certain invariable truths, which are at once plain and evident to every mind, and which are frequently made use of, in the course of geometrical reasoning. As you will frequently be obliged to refer to them, it will be well to recollect the following ones particularly:

TRUTH I.

Things which are equal to the same thing, are equal to one another.

TRUTH II.

Things which are similar to the same thing, are similar to one another.

TRUTH III.

If equals be added to equals, the wholes are equal.

TRUTH IV.

If equals be taken from equals, the remainders are equal.

TRUTH V.

The whole is greater than any one of its parts.

TRUTH VI.

The sum of all the parts is equal to the whole.

TRUTH VII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

TRUTH VIII.

Between two points only one straight line can be drawn.

TRUTH IX.

The straight line is the shortest way from one point to another.

TRUTH X.

Through one point, without a straight line, only one line can be drawn parallel to that same straight line.

SECTION I.

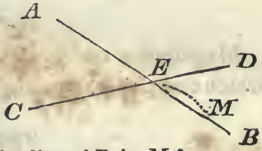
OF STRAIGHT LINES AND ANGLES.

QUERY I.

In how many points can two straight lines cut each other?

Answer. In *one* only.

Q. But could not the two straight lines AB, CD, which cut each other in the point E, have another point common; that is, could not a part of the line CD bend over and touch the line AB in M?



A. No.

Q. Why not?

A. Because there would be *two* straight lines drawn between the same points E and M, which is impossible. (Truth VIII.)

QUERY II.

If two lines have any part common, what must necessarily follow?

A. They must coincide with each other throughout, and make but one and the same straight line.

Q. How can you prove this, for instance, of the two lines CA, BM, which have the part MA common?

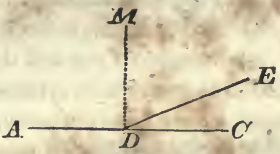
A. The common part MA belongs to the line MB as well as to the line AC, and therefore MC and AB are, in this case, but the continuation of the same straight line AM.

QUERY III.

How great is the sum of the two adjacent angles, which are formed by one straight line meeting another, taking a right angle for the measure?

A. It is equal to two right angles.

Q. How do you prove this of the two angles ADE, CDE, formed by the line ED, meeting the line AC, at the point D?

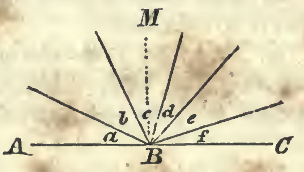


A. Because, if at D you erect the perpendicular DM, the two angles, ADE and CDE, occupy exactly the same space, as the two right angles, ADM and CDM, formed by the meeting of the perpendicular; namely, all the space on one side of the line AC. (See Truth VII.)

Q. Can you prove the same of the sum of the two adjacent angles, formed by the meeting of any other two straight lines?

QUERY IV

What is the sum of any number of angles, a, b, c, d, e, &c., formed at the same point, and on the same side of the straight line AC, taking again a right angle for the measure?

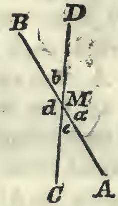


A. It is also equal to two right angles.

Q. Why?

A. Because, by erecting at the point B a perpendicular to AC, all these angles will be found to occupy the same space as the two right angles, made by the perpendicular MB.

QUERY V.

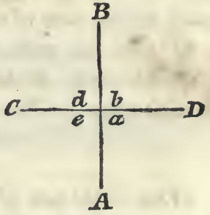


When two straight lines, AB, CD, cut each other, what relation do the angles which are opposite to each other at the vertex M, bear to each other?

A. They are equal to each other.

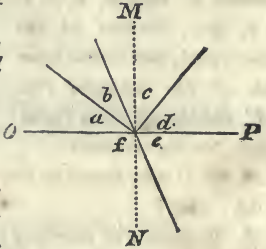
Q. How can you prove it?

A. Because, if you add the same angle *a*, first to *b*, and then to *e*, the sum will, in both cases, be the same; namely, equal to two right angles; which could not be, if the angle *b* were not equal to the angle *e* (see Truth III); and in the same manner I can prove that the two angles, *a* and *d*, are equal to each other.



Q. If the lines CD, AB, are perpendicular to each other, what remark can you make in relation to the angles d, b, e, a?

A. That each of these angles is a right angle.



Q. And what is the sum of all the angles, a, b, c, d, e, f, around the same point, equal to?

A. To four right angles.

Q. Why?

A. Because if, through that point, you draw a perpendicular to any of the lines, for instance the perpendicular MN, to the line OP, all the angles, *a*, *b*, *c*, *d*, *e*, *f*, taken together, occupy the same

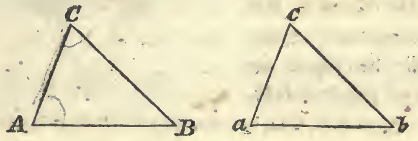
space, which is occupied by the four right angles, formed by the intersection of the two perpendiculars MN, OP.

QUERY VI.

If a triangle has one side, and the two adjacent angles, equal to one side and the two adjacent angles of another triangle, each to each, what relation do these triangles bear to each other?

A. They are equal.

Q. Supposing in this diagram the side *a b* equal to AB; the angle at *a* equal to the angle at A, and the angle at *b* equal to the angle at B; how can you prove that the triangle *a b c* is equal to the triangle ABC?



A. By applying the side *ab* to its equal AB, the side *ac* will fall upon AC, and *bc* upon BC; because the angles at *a* and A, *b* and B, are respectively equal; and as the sides *ac*, *bc*, take the same direction as the sides AC, BC, they must also meet in the same point in which the sides AC, BC, meet; that is, the point *c* will fall upon C; and the two triangles *abc*, ABC, will coincide throughout.

Q. What relation do you here discover between the equal sides and angles?

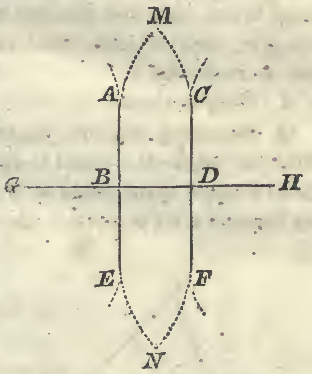
A. That the equal angles at c and C, are opposite to the equal sides ab, AB, respectively.

QUERY VII.

If two straight lines are both perpendicular to a third line, what relation must they bear to each other?

A. They must be parallel.

Q. Let us suppose the two lines AB, CD, to be both perpendicular to a third line, GH; how can you convince me that AB and CD are parallel?



A. Because, if you extend AB and CD, in the directions BE, DF, making BE and DF equal to BA and DC respectively, every thing will be equal on both sides of the line GH. Now if the lines AB, CD, are *not* parallel, they must either be converging or diverging. If they are converging, AB and CD will, when sufficiently extended, cut each other somewhere, say in M; but then (every thing being equal on both sides of the line GH) the same must take place with the lines BE, DF, on the other side of the line GH, which must cut each other somewhere in N; and there would be two straight lines cutting each other in *two* points, which is impossible. If the lines AB, CD, were diverging, BE, DF, would be the same; but it is equally impossible for two straight lines to diverge *in two directions:* consequently the two straight lines, AB, CD, can neither be converging nor diverging, and therefore they must be parallel.

Q. Can two straight lines which meet each other, be perpendicular to the same straight line?

A. No.

Q. Why not?

A. Because, if they are both perpendicular to a third line, I have just proved that they must be parallel; and if they are parallel, they cannot meet each other.

Q. From a point without a straight line, how many perpendiculars can there be drawn to that same straight line?

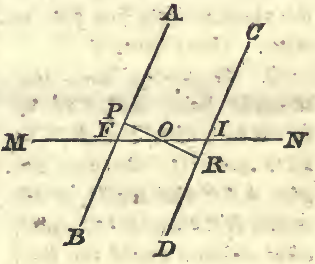
A. Only *one.*

Q. Why can there not more be drawn?

A. Because I have proved that two perpendiculars to the same straight line must be parallel to each other; and two lines, parallel to each other, cannot be drawn from one and the same point.

QUERY VIII.

If a straight line, MN, cuts two other straight lines at equal angles; that is, so as to make the angles CIN and AFN equal; what relation exists between these two lines?



A. They are parallel to each other.

Q. How can you prove it by this diagram? The line IF is bisected in O, and, from that point O, a perpendicular OP is let fall upon the line AB, and afterwards extended until, in the point R, it strikes the line CD.

A. I should first observe that the triangles OPF and ORI are equal; because the triangle OPF has a side and two adjacent angles equal to a side and two adjacent angles of the triangle ORI, each to each. (Query 6.)

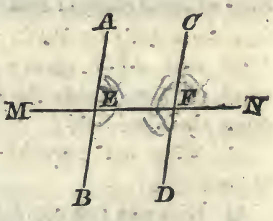
Q. Which is that side, and which are the two adjacent angles?

A. The side OI, which is equal to OF; because the point O bisects the line IF. One of the two adjacent angles is the angle IOR, which is equal to the angle FOP; because these angles are opposite at the vertex: and the other is the angle OIR, which is equal to the angle OFP; because the angle CIN, which, in the query, is *supposed* to be equal to AFN, is also equal to the angle OIR, to which it is opposite at the vertex. (Truth I.)

Q. But of what use is your proving that the triangle ORI is equal to the triangle OPF?

A. It shows that since the triangle OPF is right-angled in P, the triangle ORI must be right-angled in R; for, in equal triangles, the equal angles are opposite to the equal sides (remarks to Query 6, page 25); consequently the two lines AB, CD, are both perpendicular to the same straight line PR, and therefore parallel to each other. (Last query.)

Q. Supposing, now, two straight lines, AB, CD, to be cut by a third line, MN, so as to make the alternate angles AEF and EFD, or the angles BEF and EFC, equal, what relation would the lines AB, CD, then bear to each other?



A. They would still be parallel.

Q. How can you prove this?

A. If the angle AEF is equal to the angle EFD, the angles AEF and CFN are also equal; because EFD and CFN are opposite angles at the vertex. And, in the same manner, it may be proved, that if the angles BEF and EFC are equal, MEA and EFC are also equal; there-

fore, in both cases, there are two straight lines cut by a third line at equal angles; consequently they are parallel to each other.

Q. There is one more case, and that is: *If the two straight lines AB, CD* (in our last figure), *are cut by a third line MN, so as to make the sum of the two interior angles AEF and EFC, equal to two right angles, how are the straight lines AB, CD, then, situated with regard to each other?*

A. They are still parallel to each other. For the sum of the two adjacent angles EFC and CFN is also equal to two right angles; and therefore, by taking from each of the equal sums the common angle EFC, the two remaining angles AEF and CFN must be equal (Truth IV.); and you have again the first case, viz: two straight lines cut by a third line at equal angles.

Q. Will you now state the different cases in which two straight lines are parallel?

A. 1. *When they are cut by a third line at equal angles.*

2. *When they are cut by a third line so as to make the alternate angles equal; and,*

3. *When the sum of the two interior angles, made by the intersection of a third line, is equal to two right angles.*

QUERY IX.

Supposing the two straight lines **CD**, **EF**, *are cut by a third line* **AM** *at* unequal *angles*, **ABC**, **BHE** *(Fig. I. and II.); or so as to have the alternate angles* **CBH** *and* **BHF**, *or* **DBH** *and* **BHE**, unequal; *or in such a manner, that the sum of the two interior angles* **CBH** *and* **BHE** *(Fig. I.), or* **DBH** *and* **BHF** *(Fig. II.), is* less *than two right angles; what will then be the case with the two straight lines* **CD**, **EF**?

Fig. I.

Fig. II.

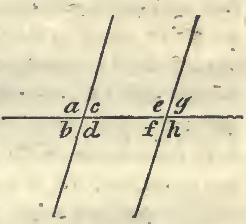
A. They will, in every one of these cases, cut each other, if sufficiently extended.

Q. How can you prove this?

A. By drawing, through the point B, another line NP at equal angles with EF, and which will then also make the alternate angles, NBH, BHF and PBH, BHE, equal, and the sum of the two interior angles, NBH and BHE, equal to two right angles; this line NP will be parallel to the line EF; consequently the line CD cannot be parallel to it; because through the point B only *one* line can be drawn parallel to the line EF. (Truth X.)

QUERY X.

Can you now tell the relation which the eight angles, a, b, c, d, e, f, g, h, formed by the intersection of two parallel lines, by a third line, bear to each other?



A. Yes. *In the first place, the angle* a *is equal to the angle* e; *the angle* c *equal to the angle* g; *the angle* b *equal to the angle* f; *and the angle* d *equal to the angle* h;—*2d. the angles* a, d, e, h, *as well as the angles* b, c, f, g, *are respectively equal to one another;—and finally, the sum of either* c *and* e, *or* d *and* f, *must make two right angles.* For if either of these cases were not true, the lines would not be parallel, (Last query.)

QUERY XI.

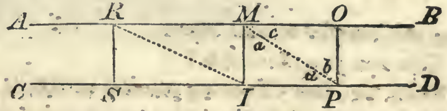
From what you have learned of the properties of parallel lines, what law do you discover respecting the distance *they keep from each other?*

A. Parallel lines remain throughout equidistant.

Q. When do you call two lines equidistant?

A. When all the perpendiculars, let fall from one line upon the other, are equal.

Q. How can you prove, that the perpendicular lines OP, MI, RS, &c. are all equal to one another?



A. By joining MP, the two triangles MPO, MPI, have the side MP common; and the angle *a* is equal to the

angle b; because a and b are alternate angles, formed by the two parallel lines MI, OP (Query 10); and the angle c is equal to the angle d; because these angles are formed in a similar manner by the parallel lines AB, CD: therefore we have a side and two adjacent angles in the triangle MPO, equal to a side and two adjacent angles in the triangle MPI; consequently these two triangles are equal; and the side OP, opposite to the angle c, in the triangle MPO, is equal to the side MI, opposite to the equal angle d, in the triangle MPI. In precisely the same manner I can prove that RS is equal to MI, and consequently also to OP; and so I might go on, and show that every perpendicular, let fall from the line AB, upon the parallel line CD, is equal to RS, MI, OP, &c. The two parallel lines AB and CD are therefore, throughout, at an equal distance from each other; and the same can be proved of other parallel lines.

QUERY XII.

If two lines are parallel to a third line, what relation do they bear to each other?

Fig. I.

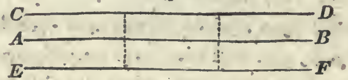
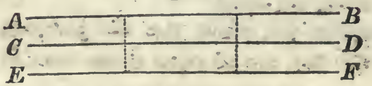


Fig. II.



They are parallel to each other.

Q. How can you prove this?

A. From the line CD being parallel to AB, it follows that every point in the line CD is at an equal distance from the line AB; and because EF is also parallel to AB,

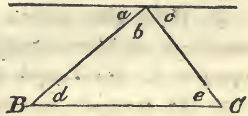
every point in the line EF is also at an equal distance from the line AB; and therefore (in *Fig. I.*) the *whole* distances between the lines CD and EF, or (in *Fig. II.*) the *differences* between the equal distances, are equal: that is, the lines CD, EF, are likewise equidistant; and consequently parallel to each other.

QUERY XIII.

What is the sum of all the angles in every triangle equal to?

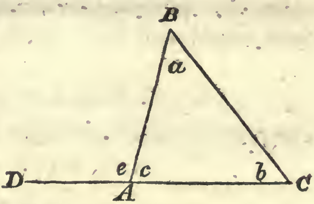
A. To two right angles.

Q. How do you prove this?



A. By drawing, through the vertex of the angle b, a straight line parallel to the basis BC, the angle a is equal to the angle d, and the angle c is equal to the angle e (Query 10); and as the sum of the three angles a, b, c, is equal to two right angles (Query 4), the sum of the three angles d, b, e, in the triangle, is also equal to two right angles.*

Q. Can you now find out the relation which the exterior angle e *bears to the two interior angles* a *and* b?



A. The exterior angle e *is equal to the sum of the two interior angles,* a *and* b.

Q. How can you prove this?

A. Because, by adding the angle c to the two angles a

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* The teacher may give his pupils an ocular demonstration of this truth, by cutting the three angles b, d, e, from a triangle, and then placing them along side of each other; they will be in a *straight* line.

and *b*, it makes with them two right angles; and by adding it to the angle *e* alone, the sum of the two angles, *e* and *c*, is also equal to two right angles (Query 3), which could not be, if the angle *e* alone were not equal to the two angles *a* and *b* together. (Truth III.)

Q. What other truths can you derive from the two which you have just now advanced?

A. 1. *The exterior angle* e *is greater than either of the interior opposite ones,* a *or* b.

2. *If two angles of a triangle are known, the third angle is also determined.*

3. *When two angles of a triangle are equal to two angles of another triangle, the third angle in the one is equal to the third angle in the other.*

4. *No triangle can contain more than one right angle.*

5. *No triangle can contain more than one obtuse angle.*

6. *No triangle can contain a right and an obtuse angle together.*

7. *In a right-angled triangle, the right angle is equal to the sum of the two other angles.*

Q. How can you convince me of the truth of each of these assertions?

RECAPITULATION OF THE TRUTHS CONTAINED IN THE FIRST SECTION.

Can you now repeat the different principles of straight lines and angles which you have learned in this section?

Ans. 1. Two straight lines can cut each other only in *one* point.

2. Two straight lines which have two points common, must coincide with each other throughout, and form but one and the same straight line.

3. The sum of the two adjacent angles, which one straight line makes with another, is equal to two right angles.

4. The sum of all the angles, made by any number of straight lines, meeting in the same point, and on the same side of a straight line, is equal to two right angles.

5. Opposite angles at the vertex are equal.

6. The sum of all the angles, made by the meeting of ever so many straight lines around the same point, is equal to four right angles.

7. When a triangle has one side and the two adjacent angles, equal to one side and the two adjacent angles in another triangle, each to each, the two triangles are equal.

8. In equal triangles the equal angles are opposite to the equal sides.

9. If two straight lines are perpendicular to a third line, they are parallel to each other.

10. If two lines are cut by a third line at equal angles, or so as to make the alternate angles equal, or so as to make the sum of the two interior angles formed by the intersection of a third line, equal to two right angles, the two lines are parallel.

11. If two lines are cut by a third line at unequal angles; or so as to have the alternate angles unequal; or in such a way as to make the sum of the two interior angles less than two right angles, these two lines will, when sufficiently extended, cut each other.

12. If two parallel lines are cut by a third line, the alternate angles are equal.

13. Parallel lines are throughout equidistant.

14. If two lines are parallel to a third line, they are parallel to each other.

15. The sum of the three angles in any triangle, is equal to two right angles.

16. If one of the sides of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.

17. The exterior angle is greater than either of the interior opposite ones.

18. If two angles of a triangle are given, the third is determined.

19. There can be but *one* right angle, or *one* obtuse angle, and never a right angle *and* obtuse angle together, in the same triangle.

20. In a right-angled triangle, the right angle is equal to the sum of the two other angles.*

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* The teacher may now ask his pupils to repeat the demonstrations of these principles.

SECTION II.

OF EQUALITY AND SIMILARITY OF TRIANGLES.

PART I.

OF THE EQUALITY OF TRIANGLES.

Preliminary Remark. There are three kinds of equality to be considered in triangles, viz: equality of area, without reference to the shape; equality of shape, without reference to the area—*similarity;* and equality of both shape and area—*coincidence.* All questions, asked in this section, will refer only to the last two kinds of equality; and those in the first part, only to the *coincidence* of triangles.

QUERY I.

If two sides and the angle which is included by them in one triangle, are equal to two sides and the angle which is included by them in another triangle, each to each, what relation do these two triangles bear to each other?

Ans. They are equal to each other in all their parts, that is, they coincide with each other throughout.

Show me that this must be the case with any two triangles, ABC, *abc*, in which we will suppose the side $AB = ab$, $AC = ac$, and the angle at A equal to the angle at *a*.

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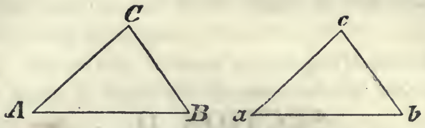


A. By placing the line *ac* upon its equal AC, the angle at *a* will coincide with the angle at A, because these two angles are equal; and the line *ab* will fall upon the line AB; and as $ab = AB$, the point *b* will fall upon B; that is, the three points of the triangle *abc* will fall upon the three points of the triangle ABC, thus:

The point *a* upon A,
" *b* " B,
" *c* " C;

consequently these two triangles must coincide.

Q. What remark can you here make with respect to the sides and angles of equal triangles?

A. The equal sides, cb, *CB, are opposite to the equal angles at* a *and A.*

QUERY II.

If one side and the two adjacent angles in one triangle, are equal to one side and the two adjacent angles in another triangle, each to each, what relation do the two triangles bear to each other?

A. They are equal, and the angles opposite to the equal sides are also equal, as has been proved in the 1st Section. (Query 6.)

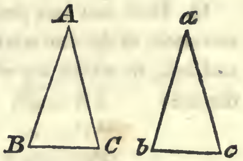
QUERY III.

What remark can you make with respect to the two angles at the basis of an isosceles triangle?

A. They are equal to each other.

Q. How can you prove it?

A. Suppose we had two equal isosceles triangles, ABC and *abc*, or, as it were, another impression, *abc*, of the triangle ABC, that is,



The side $ab = AB$,
" $ac = AC$,
" $bc = BC$.

The angle at $a =$ angle at A,
" $b =$ " B,
" $c =$ " C.

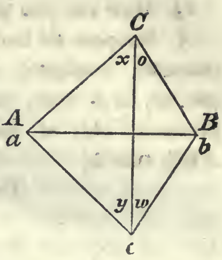
Then the sides AB, AC, *ab*, *ac*, being all equal to one another, and the angle at *a* remaining the same, whichever way we place it, the whole of the two triangles, *abc*, and ABC, will still coincide, when *abc* is placed upon ABC in such a manner that *ac* will fall upon AB, and *ab* upon AC (for you will still have two sides and the angle which is included by them in the one, equal to two sides and the angle which is included by them in the other); therefore the angle at *c* must be equal to the angle at B. And as the angle at *c* is only, as it were, another impression of the angle at C, the angles C and B must also be equal; that is, the two angles at the basis of the isosceles triangle ABC are equal: and the same can be proved of the two angles at the basis of every other isosceles triangle.

QUERY IV.

If the three sides of one triangle are equal to the three sides of another, each to each, what relation do the two triangles bear to each other?

A. They coincide with each other throughout; that is, their angles are also equal, each to each.

Q. How can you prove this, for instance, of the two triangles ABC and *abc,* in which we will suppose the side AB = *ab*,
AC = *ac*,
BC = *bc?*



That you may easier find out your demonstration, I have placed the two triangles, as you see, along side of each other, with their bases, AB and *ab*, together, and have joined their opposite vertices, C and *c* by the straight line C*c*. What do you now observe with regard to the two triangles AC*c* and BC*c ?*

A. Both are isosceles; for the sides AC and *ac*, BC and *bc*, are respectively equal; and, therefore, the angles x and y, o and w, must be equal, each to each; and as the angle x is equal to the angle y, and the angle o equal to the angle w, the sum of the two angles x and o, that is, the *whole* angle ACB, must be equal to the sum of the two angles y and w, that is, to the *whole* angle *acb;* and the two triangles, ABC, and *abc*, having two sides, AC, BC, and the angle which is included by them in the one, equal to the two sides *ac*, *bc*, and the angle which is included by them in the other, each to each, must coincide throughout, and have, consequently, all their angles respectively equal to one another. (Query 1, Sect. II.)

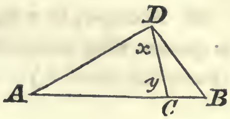
QUERY V.

Which of two angles in a triangle is greater, that which is opposite to the smaller, or that which is opposite to the greater side?

A. That which is opposite to the greater side.

Q. How can you prove it?

A. Because if in any triangle, for instance in the



triangle ABD, one side, AB, is greater than another, AD, the side AB will contain a part which is equal to AD;* and therefore, by taking upon AB the distance AC equal to AD, and joining DC, the triangle ACD will be isosceles, and the angle x will be equal to the angle y, (Query 3, Sect. II.); and as the exterior angle y must be greater than the interior opposite angle CBD, in the triangle DBC, (Query 13, Sect. I.) the angle at x will also be greater than the angle CBD; and the angle ADB being greater still than the angle x, must consequently be still more so than the angle CBD; that is, the angle ADB, opposite to the greater side AB, is greater than the angle at B, opposite to the smaller side AD: and the same can be proved of two unequal sides in any other triangle.

Q. What truth can you directly derive from this, respecting the three angles and sides of a triangle?

A. That the greatest of the three angles is opposite to the greatest of the three sides. For if the side AD, for instance, is greater than the side DB, it can be proved that the angle at B, opposite to the side AD, is greater than the angle at A, opposite to the side DB; and as the side AB is greater still than AD, the angle ADB, opposite to AB, must be greater still than the angle at B, and is therefore the greatest angle in the triangle ABD.

Q. From what you have learned of the relation which exists between the sides and angles of a triangle, can you now tell which of the sides of a right-angled triangle is the greatest?

A. Yes. That which is opposite to the right angle.

Q. Why?

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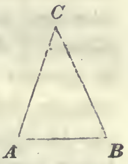
* If the magnitude A is greater than B, A must contain a *part* equal to B.

A. Because, in a right-angled triangle, the right angle is greater than either of the two other angles. (Conseq. Query 13, Sect. I.)

QUERY VI.

It has been proved before (Query 3, Sect. II.), *that in an isosceles triangle, the angles at the basis are equal: can you now prove the reverse; that is, that a triangle must be isosceles when it contains two equal angles?*

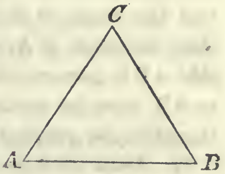


A. *Yes*. Because, if either of the two sides AC, BC, were greater than the other, the angle opposite to that side would also be greater than the angle which is opposite to the other side; but the two angles at A and B are equal, therefore the sides AC, BC, are also equal.

Q. *If the three angles in a triangle are equal to one another, what relation do the sides bear to each other?*

A. *They are also equal, and the triangle is equilateral.*

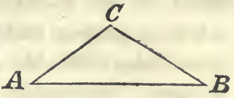
Q. How can you prove this?



A. If, in the triangle ABC, for instance, the angle at A is equal to the angle at B, I have just proved that the side BC must be equal to the side AC; and if the angle at B is also equal to the angle at C, the side AC must likewise be equal to the side AB; that is, the three sides AB, BC, AC, are equal to one another, and the triangle ABC is equilateral.

QUERY VII.

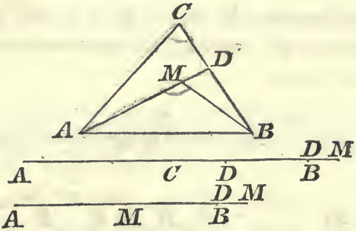
Can any one side of a triangle be greater than, or equal to, the sum of the two other sides?



A. No. A straight line being the shortest way from one point to another, it follows that, in any triangle, ABC for instance, the side AB is smaller than the sides AC and BC together.

QUERY VIII.

If, from a point M, in a triangle ABC, two lines, AM, BM, are drawn to the two extremities of any side, AB, in that triangle, what relation does the angle AMB, made by these two lines, bear to the angle ACB, which is opposite to the side AB in the triangle? And what do you observe with regard to the sum of the two lines, AM and MB, which include the angle AMB, and that of the two sides, AC, BC, of the triangle which include the angle ACB?



A. The angle AMB, made by the lines AM, BM, is always greater than the angle ACB, opposite to the side AB, in the triangle ABC; but the sum of the two lines AM, MB, is in all cases smaller than the sum of the two sides AC, CB, of the triangle.

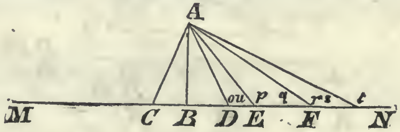
Q. How can you prove both your assertions?

A. The exterior angle MDB is greater than the interior opposite angle ACD, in the triangle ACD (Query 13, Sect. I.); and for the same reason is the exterior angle AMB greater than the interior opposite angle

MDB, in the triangle MDB; and therefore the angle AMB is greater still than the angle ACB. *2dly.* The three sides AB, AC, BC, by which the greater surface is bound, enveloping the three sides AB, AM, MB, it follows that their sum is greater than the sum of the three sides AB, AM, BM, by which the smaller surface ABM is bound; and, taking from each of the unequal sums the same line AB, which serves both as a common basis, the greater will remain where the greater was before; that is, the sum of AC and BC will still be greater than the sum of AM, BM.

QUERY IX.

If, from a point A, without a straight line MN, you let fall a perpendicular, AB, upon that line; and, at the same time, draw other lines, AD, AE, AF, &c., obliquely to different points, D, E, F, &c., in the same straight line; which is the shortest, the perpendicular, or one of the oblique lines?



A. The perpendicular is the shortest.

Q. How can you prove it?

A. Because the triangles, ABD, ABE, ABF, ABN, &c. are all right-angled in B; and in every right-angled triangle, the greatest side is opposite to the right angle. (Page 41.)

Q. And what other truths do you derive from the one you have just mentioned?

A. 1st. *The perpendicular AB measures the distance*

of the point A from the line MN; for it is the *shortest* line that can be drawn from that point to that line.

2dly. *The angles* o, p, r, t, *&c. are all obtuse,* because they are exterior angles of the right-angled triangles, ABD, ABE, ABF, &c., and, therefore, greater than the interior opposite right angle at B.

3dly. *The angles* o, p, r, t, *&c. become successively greater, and the angles* u, q, s, &c. *smaller, as the lines AD, AE, AF, &c. are drawn farther from the perpendicular.* For the exterior angle *p* is greater than the interior opposite one *o*, in the triangle ADE; the exterior angle *r* is greater than the interior opposite one *p*, in the triangle AEF; the exterior angle *t*, again, is greater than the interior opposite one *r*, in the triangle AFN; and so on.

4thly. *The oblique lines, AD, AE, AF, &c. become successively greater, as they are drawn farther from the perpendicular; that is, the line AD is greater than the line AB; the line AE than the line AD; the line AF than the line AE; and so on.* For the angles *o*, *p*, *r*, &c. are all obtuse, and become successively greater, as the triangles ADE, AEF, &c. are more remote from the perpendicular; and, therefore, the sides AE, AF, AN, &c., which are successively opposite to these angles, in the triangles ADE, AEF, AFN, must become greater with them.

5thly. *The straight lines, AC, AD, drawn on both sides of, and at an equal distance from, the perpendicular AB, are equal.* For the two triangles ABC, ABD, have the side AB common, and the side BC equal to the side BD (because the lines AC, AD, are at an equal distance from the perpendicular AB); and as the line AB is perpendicular to CD, the angle ABC, included by the sides AB, BC, in the triangle ABC, is equal to the angle ABD,

included by the sides AB, BD, in the triangle ABD; consequently, these two triangles are equal; and the third side AC in the one triangle, is equal to the third side AD in the other. (Query 1, Sect. II.)

6thly. *There is but one point in the line MN, on each side of the perpendicular, such, that a straight line, drawn from it to the point A, is of a given length.* This follows from No. 4.

7thly. *There is but one point in the line MN, on each side of the perpendicular, in which a line drawn to the point A forms with the line MN an angle of a given magnitude.* This follows from No. 3.

QUERY X.

If two sides, and the angle which is opposite to the greater of them, in one triangle, are equal to two sides ana the angle which is opposite to the greater of them in another, each to each, what relation do these two triangles bear to each other?

A. They coincide with each other in all their parts; that is, they are equal to each other.

Q. How can you prove it?

Fig. I.

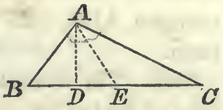
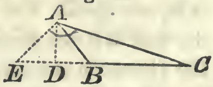


Fig. II.



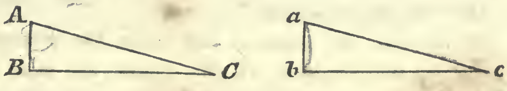
A. Because, if, in a triangle, ABC, for instance, you have the sides AB and AC, and the angle at B, which is opposite to the greater side AC, given, the whole triangle is *determined.* For, in the first place, by the angle at B, the direction of the sides AB, BC, is determined. 2*dly.* By the length of the side AB, the distance of the point A from the line BC is determined. 3*dly.* If you imagine the perpendic-

ular AD to be let fall upon BC (Fig. I.), or if the angle ABC be obtuse (as in Fig. II.) on its further extension BE, there can be but *one* point in the line BC, on this side of the perpendicular, from which a line drawn to the point A, is as long as the line AC (see consequence 6th of the preceding query); therefore, by the length of the line AC, the point C, and thereby the whole of the third line BC, is also determined.

Q. But is it not possible for the line AC to fall on the other side of the perpendicular?

A. No. Because the line AC, being greater than the line AB, would in this case be farther from the perpendicular, than the line AB (conseq. 4, preceding query), and the angle at B would then fall *without* the triangle; and because the whole triangle ABC is entirely determined, when two of its sides, and the angle which is opposite to the greater of them, are given: therefore, all triangles, in which these three things are equal, must be equal to one another.

Q. What truth can you infer from this respecting the case where the hypothenuse, and one side of a right-angled triangle, are equal to the hypothenuse and one of the sides in another right-angled triangle?



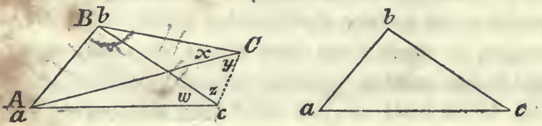
A. That these two right-angled triangles are equal to each other. For, in this case, we have two sides, and the *right* angle which is opposite to the *greater* of them, in the one, equal to two sides, and the angle which is opposite to the greater of them, in the other.

Q. But if, in Fig. II. (page 46) the two sides AC, AB, and the angle at C, opposite to the *smaller* side AB, be given, would not this be sufficient to determine the triangle ABC?

A. No. For the two lines, AB, AE, being equal, there would be two triangles, ABC and AEC possible, containing the same three things, and it would be doubtful which of the two triangles was meant.

QUERY XI.

If you have two sides, ab, bc, *of a triangle*, abc, *equal to two sides, AB, BC, of another triangle, ABC, each to each; but the angle ABC included by the two sides, AB, BC, in the triangle ABC, greater than the angle* abc, *included by the sides* ab, bc, *in the triangle* abc; *what remark can you make with regard to the two sides* ac, *AC, which are respectively opposite to those angles?*



A. *That the side* ac, *opposite to the smaller angle* abc, *in the triangle* abc, *is smaller than the side AC, opposite to the greater angle ABC, in the triangle ABC.*

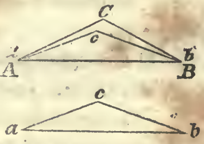
Q. How do you prove this?

A. By placing the triangle *abc* upon the triangle ABC, with the side *ab* upon AB (its equal), the side *bc* will fall *within* the angle ABC, because the angle *abc* is smaller than the angle ABC; and the extremity *c*, of the line *bc*, will either fall without the triangle ABC, as you see in the figure before you, or within it, or it may also fall upon the line AC itself.

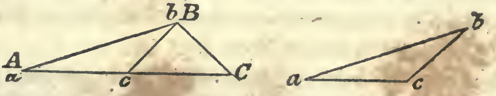
1*st*. If it falls without the triangle ABC, by imagining

the line Cc drawn, the triangle cBC will be isosceles; for we have supposed the side bc equal to BC; and because the angles at the basis of an isosceles triangle are equal (Query 3, Sect. II.), the angle z is equal to the sum of the two angles x and y; consequently greater than the angle y *alone;* and if the angle z is greater than the angle y, the two angles z and w together will be greater still than the same angle y; therefore, in the triangle ACc, the angle AcC is greater than the angle ACc; consequently the side AC, opposite to the greater angle AcC, must be greater than the side ac, opposite to the smaller angle ACc.

2dly. If the extremity of the line bc falls within the triangle ABC, the sum of the two sides ac, bc, must be smaller than the sum of the two sides AC, BC (Query 8, Sect. II.); therefore, by taking from each of these sums the equal lines bc, BC, respectively, the remainder, AC, of the greater sum (AC+BC) is greater than the remainder, ac, of the smaller sum (ac+bc).

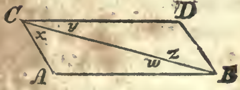


Finally. If the point c falls upon the line AC itself, it is evident that the whole line AC must be greater than its part Ac.



QUERY XII.

If, in a parallelogram, ACDB, you draw a diagonal CB, what relation do the two triangles, ABC, CDB, bear to each other?



A. They are equal to each other, and the parallelogram is divided into two equal parts.

Q. How can you prove this?

A. The two triangles, ABC and CDB, have the side CB common; and the angle y is equal to the angle w, because y and w are alternate angles, formed by the intersection of the two parallel lines CD, AB, by a third line, CB; and the angle x is equal to the angle z, because these two angles are formed in a similar manner, by the parallel lines AC, DB (Query 10, Sect. I.): and as the triangle ABC has a side CB, and the two adjacent angles, x and w, equal to the same side CB, and the two adjacent angles, z and y, in the triangle CDB, each to each; therefore these two triangles are equal (Query 6, Sect. I.), and the diagonal CB divides the parallelogram into two equal parts.

Q. What other properties of a parallelogram can you infer from the one just learned?

1st. *The opposite sides of a parallelogram are equal; that is, the side CD is equal to the side AB, and the side CA to the side DB;* for in the equal triangles, ABC, CDB, the equal sides must be opposite to the equal angles. (Conseq. of Query 1, Sect. II.)

2dly. *The opposite angles in a parallelogram are equal*; for in the two equal triangles, ABC, CDB, the same side, CB, is opposite to each of the angles, at D and A. (Conseq. of Query 6, Sect. I.)

3dly. *By one angle of a parallelogram, all four are determined;* for the sum of the four angles in a parallelogram is equal to *four right angles;* because the sum of the three angles in each of the two triangles, ABC, CDB, is equal to *two* right angles. Now, if the angle at D, for instance, is known, the angle at A is equal to it; and there remain but the two angles ACD and ABD, each

of which must be equal to half of what is wanting to complete the sum of the four right angles.

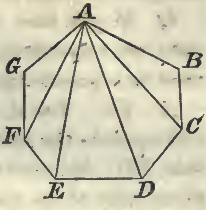
Q. If you have a quadrilateral, in which the opposite sides are respectively equal, does it follow that the figure must be a parallelogram?

A. Yes. For if, in the last figure, you have the side CD equal to the side AB, and the side AC equal to the side BD; by drawing the diagonal BC, you have the three sides of the triangle ABC, respectively, equal to the three sides of the triangle CDB; therefore, these two triangles are equal; and the angle y, opposite to the side DB, is equal to the angle w, opposite to the equal side AC; and the angle x, opposite to the side AB, is equal to the angle z, opposite to the equal side CD; that is, the alternate angles, y and w, x and z, are respectively equal: therefore the side CD is parallel to the side AB, and the side AC to the side BD, and the figure is a parallelogram.

Q. If, in a quadrilateral, you know but two sides to be equal and parallel, what will then be the name of the figure?

A. It will still be a parallelogram. For if, in the last figure, the side CD is equal and parallel to AB, by drawing the diagonal CB, you have the two sides, CB and CD, in the triangle CDB, equal to the two sides, CB, AB, in the triangle ABC, each to each; and because the side CD is parallel to the side AB, the included angle y is equal to the included angle w; therefore the two triangles are equal (Query 1, Sect. II.), and the side AC is also equal and parallel to the side DB, as before.

QUERY XIII.



If, from one of the vertices of a rectilinear figure, diagonals are drawn to all the other vertices, into how many triangles will this rectilinear figure be divided?

A. Into as many as the figure has sides less two. For it is evident, that if, from the vertex A, for instance, you draw the diagonals AF, AE, AD, AC, to the vertices F, E, D, C, each of the two triangles AGF, ABC, will need for its formation two sides of the figure, and a diagonal; but then every *one* remaining side of the figure will, together with two diagonals, form a triangle; therefore there will be as many triangles formed, as there are sides less the two, which are additionally employed in the formation of the two triangles AGF, ABC.

Q. And what is the sum of all the angles, BAG, AGF, GFE, FED, EDC, DCB, CBA, equal to?

A. To as many times two right angles as the figure ABCDEFG has sides less two. For as every rectilinear figure can be divided into as many triangles as there are sides less two; and because the sum of the three angles in each triangle is equal to two right angles (Query 13, Sect. I.) there will be as many times *two right angles* in all the angles of your figure, as there are triangles; that is, as many as the figure has sides less two.

SECTION II.

PART II.

OF GEOMETRICAL PROPORTIONS,* AND SIMILARITY OF TRIANGLES.

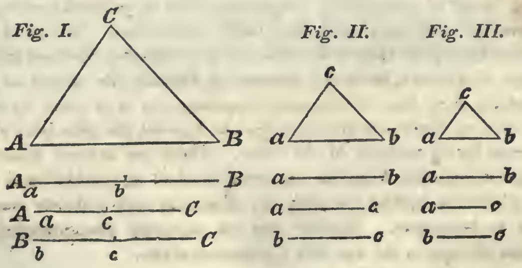
WHENEVER we compare two things with regard to their magnitude, and inquire *how many times* one is greater than the other, we determine the *ratio* which these two things bear to each other. If, in this way, we find out that the one is two, three, four, &c. times greater than the other, we say that these things are *in the ratio of one to two, to three, to four, &c.: e. g.* If you compare the fortunes of two persons, one of whom is worth $10,000, and the other $20,000, you say, that their fortunes are in the ratio of one to two. Or if you compare two lines, one of which is two, and the other six feet long, you say of these lines, that they are in the

* It is the design of the author to give here a perfectly *elementary* theory of geometrical proportions, and to establish every principle *geometrically*, and by simple *induction*. Intending the above theory for those who have not yet acquired the least knowledge of Algebra, he is not allowed to identify the theory of proportions with that of algebraic equations (as it is done by some writers on Mathematics), and then to find out the principles of the former by an analysis of the latter. There are several disadvantages inseparable from the algebraic method of considering a ratio as a fraction, besides the difficulty of making such a theory accessible to beginners. Neither can an algebraic demonstration be made obvious to the eye like a geometrical one.

ratio of one to three, because the second line is three times as long as the first.

It frequently occurs, that two things are to each other in the same ratio in which two others are; we then say that these things are in *proportion.* This is frequently the case in the fine arts; but particularly in the science of Geometry, from which these proportions are called *geometrical.* To give an example: If you draw a house, you must draw it *according to a certain scale;* that is, you must draw it one thousand, two thousand, three thousand, &c. times smaller than the building itself: but then you are obliged to reduce every part of it *in proportion.* If, for instance, you draw the front of the house one thousand times smaller than the original, you must reduce the windows, doors, and every other part, *in the same ratio.* If, on the contrary, the windows were reduced *two* thousand times, whilst the doors and other parts were reduced only *one* thousand times, your picture would be *out of proportion*, because the different parts would be reduced by *different* ratios. In this case your picture would be distorted; and would *not resemble the original.*

The same is the case with resemblance, produced in any other kind of drawings; but particularly in geometrical figures.



If the two triangles, ABC, *abc*, are to be similar to each other, it is necessary that they should be constructed after the same manner, and that the side AC should be exactly as many times greater than the side *ac*, as the side BC is greater than *bc*, and the side AB than *ab*. If (Fig. I. and II.) the side AB, for instance, is twice as great as the side *ab;* that is, if the side *ab* is half of the side AB; the side *ac* must also be half of the side AC, and the side *bc* half of the side BC; that is, the three sides, *ab*, *ac*, *bc*, of the triangle *abc*, must be in *proportion* to the three sides, AB, AC, BC, of the triangle ABC. Again, if (Fig. I. and III.) the side AB is three times as great as the side *ab;* that is, if the side *ab* is one third of the side AB; the side *ac* must also be one third of the side AC, and the side *bc* one third of the side BC; or the triangles *abc*, ABC, would not be similar to each other. The same holds true of all other geometrical figures, composed of any number of sides. If they are similar, their sides are *proportional* to each other.

There are different ways of denoting a geometrical proportion. Some mathematicians express the proportionality of the sides, *ab*, *ac*, of the triangle *abc* (Fig. II.), to the sides AB, AC, of the triangle ABC (Fig. I.), in the following manner:

$$AB : ab :: AC : ac;$$

or,

$$AB \div ab :: AC \div ac;$$

and also

$$AB : ab = AC : ac,^*$$

which is read thus:

AB is to *ab*, as AC is to *ac*.

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* The first manner of expressing a proportion is now in general use among the English and French mathematicians; the second is sometimes met with in old English writers, and the third way is adopted in Germany.

As a proportion is nothing less than the equality of two ratios, the third way of denoting a proportion, in which the sign of equality is put between the two ratios, seems to be the most natural. The reason why the sign of division (see Notation and Significations), is put between the two terms, AB, *ab*, of a ratio, is obvious; for a ratio points out how many times one term (the side *ab*) is contained in the other (the side AB).

The first and fourth terms of a proportion, together, are called *extremes;* because one of them stands at the *beginning*, and the other at the *end*, of a proportion: the second and third terms, standing in the *middle*, are, together, called the *means*.

The following principles of geometrical proportions ought to be well understood and remembered:

1st. *It is important to observe, that in every geometrical proportion the two ratios may be inverted;* that is, instead of saying,

$$\mathrm{AB} : ab = \mathrm{AC} : ac,$$

you may say,

$$ab : \mathrm{AB} = ac : \mathrm{AC};$$

for, the order of terms being changed in *both* ratios, they are still equal to one another; but, leaving one ratio unaltered, if you change the order of terms in the other, the proportion will be destroyed. You cannot say,

$$ab : \mathrm{AB} = \mathrm{AC} : ac;$$

for the *smaller* side, *ab*, is contained *twice* in the greater side, AB (Fig. I. and II.); but the *greater* side, AC, is not contained *once* in the smaller side, *ac*.

2d. Another remarkable property of geometrical proportions is, that *you may change the order of the means, or extremes, without destroying the proportion.* Thus you may change the proportion

$$AB : ab = AC : ac \quad . \; . \; . \; . \; . \quad \text{(I.)}$$

into

$$AB : AC = ab : ac \quad . \; . \; . \; . \; . \; . \quad \text{(II.)}$$

or by changing the extremes into

$$ac : ab = AC : AB \quad . \; . \; . \; . \; . \quad \text{(III.)}$$

The reason why you have a right to do this, is easily comprehended. If, in the first proportion, the side AB is as many times greater than *ab*, as AC is greater than *ac*, the ratio of AB to AC will be the same as that of *ab* to *ac*. In Fig. I. and II. (page 54), we have *ab* equal to one half of AB; consequently *ac* is also equal to one half of AC; and, therefore, let the ratio of the two lines, AB to AC, be whatever it may, their halves, *ab* and *ac*, must be in the same ratio. To give another example: If A's garden is five times greater than B's, *half* of A's garden is also five times greater than half of B's garden. The second proportion (II.) would still be correct, if, as in Fig. I. and III., the sides AB, AC, were *three* times as great as the sides *ab*, *ac;* for then the thirds of AB and AC would still be in the same proportion as the whole lines AB and AC. Nothing can now be easier than to extend this mode of reasoning, and show the generality of the principle here advanced. The correctness of the third proportion might be proved precisely in the same manner as that of the second; for the third proportion (III.) differs from the second (II.) only *in the order* in which the two ratios are placed; and of two equal things, it does not matter which you put first. The correctness of the second proportion proves, therefore, that of the third proportion.

3d. *If you have two geometrical proportions, which have one ratio common, the two remaining ratios will, again, make a proportion;* for if two ratios are equal to the same ratio, they must be equal to each other. (See Axioms, Truth I.) If you have the two proportions

$$AB : ab = AC : ac$$
$$AB : ab = BC : bc$$

you will also have the proportion

$$AC : ac = BC : bc.$$

For an *illustration* of this principle, we may take the two triangles ABC, *abc* (Fig. I. and II.): If the sides AB and *ab* are in proportion to the sides AC and *ac*, and also in proportion to the sides BC and *bc*, the *three* sides of the triangle ABC will be in proportion to the three sides of the triangle *abc;* therefore, any *two* sides of the first triangle will be in proportion to the two corresponding sides of the other triangle.

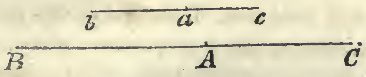
4th. Another important principle of geometrical proportions is this: *If you have several geometrical proportions, of which the second has a ratio common with the first, the third a ratio common with the second, the fourth a ratio common with the third, and so on; the sum of all the first terms of these proportions will bear the same ratio to the sum of all the second terms, which the sum of all the third terms does to the sum of all the fourth terms, that is, the sums will again make a proportion.*

To prove this, we will, in the first place, consider the simplest case; that of *two* proportions only; and, the easier to comprehend it, take the same two proportions which we have just had under consideration, viz:

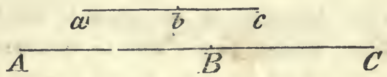
$$ac : AC = ab : AB$$
$$ab : AB = bc : BC.$$

We know, from the two triangles, ABC and *abc* (Fig. I. and II.), that, in the first proportion, *ac* is half of AC; consequently *ab* is also half of AB, and, in the second proportion, *bc* is also half of BC. Thus, each of the two first terms, *ab*, *ac*, is half of its second term; and consequently each of the third terms, *bc*, *ab*, is half of its corresponding fourth term; therefore, adding *ab* and *ac*

together, their *sum* will be one half of the sum of AB and AC; and so will *bc* and *ab*, be, together, one half of the sum of BC and AB. For the sake of illustration, you may measure off the length of *ab* and *ac*, upon



the line *bc*, and the length of AB and AC on another line BC; and you will find that the line *bc* is exactly one half of the line BC. For the line *bc*, composed of two parts, *ab*, *ac*, each measuring exactly one half of the corresponding two parts, AB, AC, of which the line BC is composed, must evidently be one half of the *whole* line BC. In the same way you may convince yourself that



the line *ac*, composed of the two parts, *ab* and *bc*, measures one half of the second line AC, composed of the two parts AB and BC: and therefore you will have

$$\overline{ab+ac} : \overline{AB+AC} = \overline{bc+ab} : \overline{BC+AB}.*$$

Although, in our example, we have chosen a proportion in which the first and third terms are exactly one half of the second and fourth terms, yet it is easy to perceive, that the same course of reasoning will apply to any other two proportions. Thus, if the first terms in the above proportions were one third, or one fourth, or one fifth,

* The lines over $ab+ac$, $AB+AC$, &c., mark that $ab+ac$, $AB+AC$, &c., are but single lines composed of the two parts, *ab*, *ac*, and AB, AC.

&c., of the corresponding second terms, the *sum* of all the first terms would also be one third, or one fourth, or one fifth, &c., of the sum of all the second terms; and the same would be the case with regard to the sum of the third and fourth terms. It is also evident, that our principle would still hold true, if, instead of two proportions, we had three, four, or more proportions given, of which two and two have a common ratio. If, for instance, we had the *three* proportions

$$ac : AC = ab : AB$$
$$ab : AB = bc : BC$$
$$bc : BC = ac : AC,$$

we should, according to our principle, have

$$\overline{bc + ab + ac} : \overline{BC + AB + AC} = \overline{ac + bc + ab} : \overline{AC + BC + AB}.$$

Each of the three lines, *bc*, *ab*, *ac*, would be one half of its corresponding second term; and in the same way would each of the three lines, *ac*, *bc*, *ab*, be one half of its corresponding fourth term; and, therefore, the *sum* of the three lines, *bc*, *ab*, *ac*, or, which is the same, a line as great as the three lines, *bc*, *ab*, *ac*, together, would be one half of the sum of the three lines, BC, AB, AC, or a line as great as the three lines, BC, AB, AC, together; and the same would be the case with the sum of the third and fourth terms. And in like manner can this principle be extended to four, five, six, and more proportions.

5th. Another principle, which it is important to recollect, is, that *by adding the second term of a proportion once, or any number of times, to the first term, and the fourth term the same number of times to the third term, you will still have a proportion.* To give an example: In the proportion

$$\mathrm{AB} : ab = \mathrm{AC} : ac,$$

let there be added the second term *ab*, in the first place, *once* to the first term AB; and the fourth term *ac* also *once* to the third term AC. Our proportion is then changed into

$$\overline{\mathrm{AB} + ab} : ab = \overline{\mathrm{AC} + ac} : ac,$$

in which the first term, $\overline{\mathrm{AB} + ab}$, instead of being only twice as great as *ab*, is now, by the addition of the term *ab* itself, three times as great as *ab*; and for the same reason is $\overline{\mathrm{AC} + ac}$ three times as great as *ac*. The two new ratios,

$$\overline{\mathrm{AB} + ab} : ab, \text{ and}$$

$$\overline{\mathrm{AC} + ac} : ac,$$

are therefore equal, and consequently make a proportion. The same would be the case, if, instead of adding the second and fourth terms *once*, you would add them *twice* respectively to the first and third terms; with the only difference, that the first term, $\overline{\mathrm{AB} + 2ab}$, would then be four times as great as the second term *ab*. A similar change would take place with regard to the third term, $\overline{\mathrm{AC} + 2ac}$, which would then be four times as great as the term *ac*; and you would have the proportion

$$\overline{\mathrm{AB} + 2ab} : ab = \overline{\mathrm{AC} + 2ac} : ac.$$

If the second term were added *three* times to the first term; the first term, $\overline{\mathrm{AB} + 3ab}$, would be five times as great as *ab*; and the third term, $\overline{\mathrm{AC} + 3ac}$, would also be five times as great as *ac*; and so on.

In precisely the same manner you may prove that, *by adding the first term once, or any number of times, to the second term, and the third term the same number of times to the fourth term, the result will still be a proportion.* Thus, our proportion

$$AB : ab = AC : ac,$$

may be changed into

$$AB : \overline{ab + AB} = AC : \overline{ac + AC},$$

or into

$$AB : \overline{ab + 2AB} = AC : \overline{ac + 2AC}, \&c.$$

It is also evident that the same principle will hold of any other geometrical proportion.*

6th. *For the same reason that the second term of a geometrical proportion may be* added *once or any number of times to the first term, and the fourth term the same number of times to the third term, without destroying the proportion; the second term may also be* subtracted *once or any number of times from the first term, provided the fourth term is the same number of times subtracted from the third term, and the result will still be a proportion.*

If, in the geometrical proportion

$$AB : ab = AC : ac,$$

the first term (AB) is twice as great as *ab*, and AC twice as great as *ac*, we shall, by subtracting *ab* from AB, and *ac* from AC, make the two terms in each ratio equal; and we shall have a new proportion,

$$AB - ab : ab = AC - ac : ac.$$

---

* The teacher had better show this to the pupil, particularly as the above mode of demonstrating this principle admits of an ocular demonstration by measurements. For if the teacher uses *lines* for the terms of his proportions, and not abstract numbers, which are always more difficult to be comprehended, he can actually perform these additions, by extending the line AB, for instance, to once or twice the length of the line *ab*, and then show, by measuring these lines, that the first term is really as many times greater than the second term, as the third term is greater than the fourth term. In this manner the demonstrations will not only be perfectly geometrical, but also have the advantage of the inductive method.

If AB were three times as great as ab, AC would, of sourse, be three times as great as ac; and therefore, by subtracting ab from AB, and ac from AC, the first term $(AB - ab)$, in the last proportion; would be twice as great as ab; and for the same reason would $AC - ac$, be twice as great as ac. In the same manner may this principle be applied to every other geometrical proportion; and it may also be proved, that, *by subtracting the first term of a geometrical proportion once or any number of times from the second term, and the third term the same number of times from the fourth term, the proportion will not be destroyed.*

7th. *If all the terms of a geometrical proportion are multiplied or divided by the same number, the proportion remains the same.*

For an example, we will again take the proportion

$$AB : ab = AC : ac,$$

in which ab is half of AB, and ac half of AC. Then it is evident, that a line, which is, for instance, ten times as long as ab, that is, a line which contains the line ab ten times, is still half of a line which contains the line AB ten times; and in like manner is a line ten times as long as ac still half of a line ten times as long as AC; consequently the proportion

$$10\,AB : 10\,ab = 10\,AC : 10\,ac$$

is the same as

$$AB : ab = AC : ac,$$

because in both proportions the first term in each ratio is double of the second term.

Neither would our proportion change, if, instead of multiplying each term by 10, we were to multiply it by 2, by 3, by 4, &c., or even by fractions; for the reasoning would, in every one of these cases, be precisely the same as in the case of our multiplying by ten.

It is also easy to apply the same principle to any other geometrical proportion.

If, instead of *multiplying* each term of the proportion

$$AB : ab = AC : ac,$$

we *divide* it by ten, it is evident that the tenth part of the line ab will still be half of the tenth part of the line AB; and so will the tenth part of the line ac be half of the tenth part of the line AC; consequently the proportion

$$\tfrac{1}{10}AB : \tfrac{1}{10}ab = \tfrac{1}{10}AC : \tfrac{1}{10}ac$$

is still the same as

$$AB : ab = AC : ac;$$

and the same reasoning may be applied to the division by any other number, and to any other geometrical proportion.

8th. *If three terms of a proportion be given, the fourth term can easily be found.* Let there be the three terms of a proportion,

$$AB : ab = AC :$$

to which the fourth term is wanting. Then, by knowing how many times the line ab is smaller than the line AB, or, which is the same, whatever part of the line AB the line ab is, you can easily take the same part of the line AC, which will be the fourth term of your proportion. If you know, for instance, that the line ab is one half of the line AB, you would at once conclude, that the required fourth term in your proportion must be one half of the line AC: this is, as we know, really the case with our proportion, where the fourth term ac, which we supposed here to be unknown, is really one half of AC. If ab were one third of AB, you would conclude that your fourth term must be one third of AC; and so on. If, instead of the fourth term, another, for instance the

second term, were unknown, you could find it in a manner similar to the one just given. For, one ratio being expressed, you will always know the relation which the term you are to find must bear to the term with which it is to form a ratio.

9th. *Geometrical proportions are also frequently made use of in common Arithmetic, and in Algebra.* You can say of the two numbers 3 and 6, that they are in proportion to the numbers 4 and 8; because 3 are as many times contained in 6, as 4 in 8, which may be expressed thus:

$$3 : 6 = 4 : 8.$$

For this reason, if four lines are in a geometrical proportion, their length, expressed in numbers of rods, feet, &c., will be in the same proportion.

10th. *It is to be remarked, that in every geometrical proportion, expressed in numbers,* the product obtained by multiplying the two mean terms together, is equal to the product obtained by multiplying the two extreme terms.* In the above proportion, $3 : 6 = 4 : 8$, for instance, we have 3 times 8 equal to 6 times 4. For, the first of our extreme terms, 3, being exactly as many times *smaller* than the first of our mean terms, 6, as the second of our extreme terms, 8, is *greater* than the second of our mean terms, 4 (namely, twice); what the multiplier 3, in the one case, is *smaller* than the multiplier 6, in the other, is made up by the multiplicand 8, which is as many times *greater* than the multiplicand 4, as the multiplier 3 is smaller than the multiplier 6; and in a similar manner

* For we cannot multiply *lines* together, but merely the *abstract numbers*, which express their relative length.

we could prove the same of any other geometrical proportion. To give but one more example: In the proportion

$$2 : 4 = 3 : 6,$$

we have, again, twice 6 equal to 4 times 3; because the first multiplier, 2, is exactly as many times *smaller* than the second multiplier, 4, as the first multiplicand, 6, is *greater* than the second multiplicand 3 (namely, twice).*

If both ratios of our proportion were inverted, as, for instance, $4 : 2 = 6 : 3$, our principle would still prove to be correct. For we have again 4 times 3 equal to twice 6. The only difference consists in the mean terms having now become the extreme terms, and *vice versa.* If we change the order of the means and extremes, their products remain still the same. For 3 times 8 are the same as 8 times 3; and 6 times 2 the same as twice 6.

When, in a geometrical proportion, the two mean terms are equal to one another, either of them is called a *mean proportional* between the two extremes. Thus, in the proportion

$$4 : 6 = 6 : 9,$$

6 is a mean proportional between 4 and 9.

What you have learned of geometrical proportions will enable you to understand every principle in plane Geometry; we will therefore continue our inquiries into the principles of geometrical figures.

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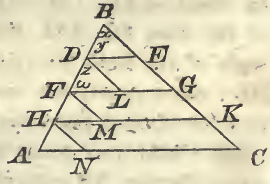
* The teacher may illustrate this principle by a balance; showing that 2 weights, of 6 pounds each, are in equilibrium with 4 weights of 3 pounds each. The weights in this example, 6 pounds and 3 pounds, are the multiplicands, and their number 2 and 4 are the respective multipliers.

QUERY XIV.



If you divide one side, AB, of a triangle, ABC, into any number of equal parts, for instance four, and then, from the points of division D, F, H, draw the lines DE, FG, HK, parallel to the side AC, what remark can you make with regard to the other side BC?

A. That the other side, BC, is divided into as many equal parts as the side AB.

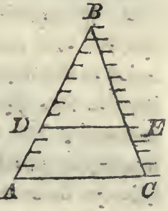
Q. How can you prove this?

A. By drawing the lines DL, FM, HN, parallel to the side BC, the triangles, BDE, DFL, FHM, HAN, are all equal to one another. For, comparing, in the first place, the two triangles, BDE, DFL, we see that the side BD is equal to DF (because we have divided the line AB into equal parts); and the angle x is equal to the angle z, because these angles are formed by the two parallels DL and BC, being intersected by the straight line AB (Query 10, Sect I.); and the angle y is equal to the angle w, because y and w are formed, in a similar manner, by the two parallels, DE, FG, being intersected by the same straight line AB; consequently we have one side, DB, and the two adjacent angles x and y, in the triangle BDE, equal to one side DF, and the two adjacent angles, z and w, in the triangle DFL; therefore these two triangles are equal to each other (Query 6, Sect I.); and the side DL, opposite to the angle w, in the triangle DFL, is equal to the side BE, opposite to the equal angle y, in the triangle BDE; and in the same manner it can be proved, that FM and HN, are also equal to BE. Now, each of the quadrilaterals, DELG, FGMK, HKNC, is a parallelo-

gram (because the opposite sides are parallel); and as the opposite sides of a parallelogram are equal, DL must be equal to EG, FM to GK, and HN to KC. But each of the lines DL, FM, HN, is equal to BE; therefore, each of the lines EG, GK, KC, must also be equal to BE; consequently the line BC is divided into the same number of equal parts as the line AB.

Q. Could you prove the same principle in the case where the line AB is divided into five, six, or more equal parts?

QUERY XV.



If, in a triangle, ABC, you draw a line, DE, parallel to one of the sides, say AC; what relation do the parts BD, DA; BE, EC, into which the sides AB and BC are divided, bear to each other, and to the whole of the sides AB, BC?

A. The upper parts, BD and BE, as well as the lower parts, DA and EC, are in the same ratio, in which the whole sides AB, BC, themselves are.

Q. Why?

A. Because you can imagine the side AB to be successively divided into smaller and smaller parts, until one of the points of division shall have fallen upon the point D: then, by drawing, through all the points of division, parallel lines to the side AC, the side BC will be divided into as many equal parts as the side BA (last Query); and as the line DE itself will be one of these parallels, BE will have as many of these parts marked as BD; and EC as many as DA: and therefore the ratio of

the *whole* of the line BA to the *whole* of the line BC, must be the same as that of BD to BE, or DA to EC.

Q. How can you express these proportions in writing?

A.
$$BA : BC = BD : BE$$
$$BA : BC = DA : EC;$$
consequently, also,
$$BD : BE = DA : EC$$
(3d principle of proportion).

Q. Is the reverse of the same principle also true? that is, must the line DE be parallel to AC, when the parts BD and BE, and DA and EC, are proportional to each other, or to the whole of the sides BA, BC?

A. Yes. For you need only imagine the side BA to be again successively divided into smaller and smaller parts, until one of the points of division shall have fallen upon D. Then, it is evident that, by drawing, as before, through the points of division, parallels to the side AC, DE itself must be one of them, if BE shall again have as many of these parts marked as BD, and EC as many as DA; for only in this case can BE, BD, and EC, DA, be proportional to each other, and to the whole of the sides BC and BA.

Remark. It has already been stated (page 55), that two geometrical figures cannot be similar to each other, unless they are constructed after the same manner, and have their sides proportional. We will now give the strictly geometrical definition of the same principle for *rectilinear* figures.

In order that two rectilinear figures may be similar to each other, it is necessary,

1st, *That both figures should be composed of the same number of sides;**

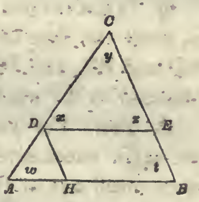
* This will, of course, always be the case in triangles.

2dly, *That the angles in one figure should be equal to the angles in the other, each to each ;*

3dly, *That these angles should follow each other in precisely the same order in both figures ; and,*

4thly, *That the sides, which include the equal angles in both figures (and which are therefore called the corresponding or homologous sides*), should be in a geometrical proportion.*

QUERY XVI.



If, in a triangle, ABC, you draw a line, DE, parallel to one of the sides, say AB, what relation does the triangle DEC, which is cut off, bear to the whole of the triangle ABC?

A. The triangle DEC is similar to the triangle ABC.

Q. Why?

A. Because the three angles, x, y, z, of the triangle DEC, are equal to the three angles, w, y, t, of the triangle ABC, each to each; for the angles x and z are respectively equal to the angles w, t; because the line DE is parallel to AB (Query 10, Sect. I.). This satisfies the three first conditions of similarity. Moreover, we have the proportion CD : CE = CA : CB (preceding Query), and by drawing DH parallel to the side CB, also the proportion CD : BH (or ED) = AC : AB; therefore, the three sides of the triangle DEC are proportional to the three sides of the triangle ABC, which is the fourth condition of similarity: consequently these two triangles are similar to each other.

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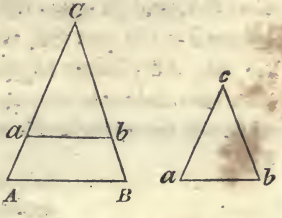
* In triangles, the corresponding sides are those which are opposite to the equal angles.

QUERY XVII.

If the three angles in one triangle are equal to the three angles in another triangle, each to each, what relation do these triangles bear to each other?



A. They are similar.

Q. How can you prove it?

A. By applying the triangle *abc* to the triangle ABC, the angle at *c* will coincide with the angle at C, and the side *ca* will fall upon CA, and *cb* upon CB; and as the angles at *a* and *b*, in the triangle *abc*, are respectively equal to the angles at A and B, in the triangle ABC, the side *ab* will fall parallel to the side AB (Query 8, Sect. I.) and we shall have the same case as in the preceding Query: consequently, the triangles *abc* and ABC will be similar to each other.

Q. Supposing you have a triangle, of which you know only two angles, respectively, equal to two angles in another triangle, what can you infer with regard to these two triangles?

A. That they must still be similar to each other. For two angles of a triangle always determine the third one (page 34, 2d.).

QUERY XVIII.

If you have two triangles, abc, *ABC* (see the last figure), *and only know that one angle at* c, *in the one, is equal to one angle at C in the other, but that the sides, which include that angle in both triangles, are in a geometrical proportion, what inference can you draw from it?*

A. That these triangles are again similar to each

other. For if you imagine the triangle *abc* placed, as before, upon the triangle ABC, the angle at *c* will again coincide with the angle at C, and the side *ca* will fall upon CA, and *cb* upon CB; and as *ca* and *cb* are proportional to CA and CB, the side *ab* will fall parallel to the side AB (Query 15, Sect. II.); and we shall once more have the same case as in Query 16, Sect. II.; consequently the triangle *abc* will be similar to the triangle ABC.

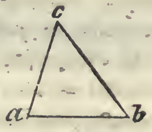
QUERY XIX.

Let us now consider the case, where all the angles of two triangles are unknown; but the three sides of the one are in proportion to the three sides of the other; what relation will these triangles bear to each other?

A. *They will still be similar to each other?*

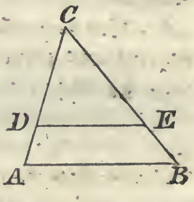
Q. How can you prove it?

A. Let us suppose, for instance, that the three sides of the triangle *abc* are in proportion to the three sides of the triangle ABC; that is, let us have the proportions



$$ac : ab = AC : AB$$
$$ac : cb = AC : CB.$$

Then make CD equal to *ca*, and draw through the point D the line DE, parallel to AB; and the triangle CDE will be similar to the triangle CAB (Query 16, Sect. II.), and we shall have the proportions



$$DC : DE = AC : AB$$
$$DC : CE = AC : CB,$$

in which the two ratios, AC : AB, and AC : CB, are the same as in the first two proportions; consequently, com-

paring these two proportions with the two preceding ones, we shall have

$$DC : DE = ac : ab$$
$$DC : CE = ac : cb$$

(see theory of proportions, principle 3d).

Now, as I have made DC equal to *ac*, I can write *ac* instead of DC, in the two last proportions; and they will then become

$$ac : DE = ac : ab,$$
$$ac : CE = ac : cb.$$

The upper one expresses, that the line DE is as many times greater than *ac*, as the line *ab* is greater than the same line *ac* (Definition of geometrical proportions); consequently the line DE is equal to the line *ab*. In like manner does the lower one express, that CE is as many times greater than *ac*, as *cb* is greater than the same line *ac*; consequently CE is also equal to *cb*; and the three sides of the triangle DEC, are equal to the three sides of the triangle *abc*, each to each; therefore these two triangles are equal to one another (Query 4, Sect. II.); and as the triangle DEC is similar to the triangle ABC, the triangle *abc* will also be similar to it.

*   *   *

Q. Will you now briefly state the different cases, in which two triangles are similar to one another?

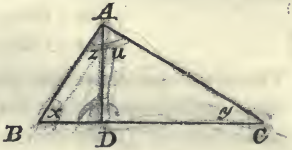
A. 1st. *When the three angles in one triangle are equal to the three angles in another, each to each; and also when* two *angles in one triangle are equal to two angles in another, each to each; because then the third angle in the one is also equal to the third angle in the other.*

2dly. *When an angle in one triangle is equal to an angle in another, and the two sides which include that*

angle in the one triangle, are in proportion to the two sides which include that angle in the other triangle.

3dly. *When the three sides of one triangle are in proportion to the three sides of another.*

QUERY XX.



If you have a right-angled triangle, ABC, and from the vertex A, of the right angle, let fall a perpendicular, AD, upon the hypothenuse BC, what relation do the two triangles, ABD and ACD, into which the whole triangle is divided, bear to each other, and to the whole triangle ABC itself?

A. The two triangles, ABD and ACD, are similar to each other, and to the whole triangle ABC.

Q. How can you prove this ?

A. The triangle ABD is similar to the whole triangle ABC, because the two triangles being both *right-angled*, and having the angle at x common, have two angles in one triangle, respectively, equal to two angles in the other (page 73, case 1st); and for the same reason is the triangle ACD similar to the whole triangle ABC (both being *right-angled*, and having the angle y common); and as each of the two triangles, ABD, ACD, is similar to the whole triangle ABC, these two triangles must be similar to each other. (Truth II.)

Q. What important inferences can you draw from the principle you have just established?

A. 1st. *In the two similar triangles, ABD and ACD, the sides which are opposite to the equal angles, must be*

in proportion (condition 4th of geometrical similarity, page 70); and we shall therefore have the proportion

$$BD : AD = AD : DC;^{*}$$

that is, *the perpendicular AD is a mean proportional between the two parts into which it divides the hypothenuse.* (Theory of proportion, page 66.)

2*dly*. From the two similar triangles, ABC, ABD, we shall have the proportion

$$BD : AB = AB : BC;$$

that is, *the side AB, of the right-angled triangle ABC, is a mean proportional between the whole hypothenuse BC, and the part BD, cut off from it by the perpendicular AD.*†

3*dly*. The two similar triangles, ACD and ABC, give the proportion

$$DC : AC = AC : BC;$$

that is, *the other side, AC, of the right-angled triangle ACD, is also a mean proportional between the whole hypothenuse and the other part, DC, cut off from it by the perpendicular AD.*

Remark. The five last queries comprise one of the most important parts of Geometry. The principles contained in them are applied to the solution of almost every geometrical problem. The beginner will therefore do well to render himself perfectly familiar with them.

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* The first ratio is formed by the two sides, BD and AD, of the triangle ADB, of which BD is opposite to the angle z, and AD to the angle x; and the second ratio is formed by the two corresponding sides, AD, DC, of the triangle ADC; because the sides AD, DC, are opposite to the angles y and u, which are respectively equal to z and x.

† The part BD of the hypothenuse, situated between the extremity B of the side AB, and the foot D of the perpendicular AD, is sometimes called the *adjacent segment* to AB. (Legendre's Geometry, translated by Professor Farrar.)

RECAPITULATION OF THE TRUTHS CONTAINED IN THE SECOND SECTION.

PART I.

Ques. Can you now repeat the different principles respecting the equality and similarity of triangles, which you have learned in this section?

Ans. 1. If, in two triangles, two sides of the one are equal to two sides of the other, each to each, and the angles which are included by them are also equal to one another, the two triangles are equal in all their parts, that is, they coincide with each other throughout.

2. In equal triangles, that is, in triangles which coincide with each other, the equal sides are opposite to the equal angles.

3. If one side and the two adjacent angles in one triangle are equal to one side and the two adjacent angles in another triangle, each to each, the two triangles are equal, and the angles opposite to the equal sides are also equal.*

4. The two angles at the basis of an isosceles triangle are equal to one another.

5. If the three sides of one triangle are equal to the three sides of another, each to each, the two triangles coincide with each other throughout; that is, their angles are also equal, each to each.

6. In every triangle, the greater side is opposite to the greater angle, and the greatest side to the greatest angle.

7. In a right-angled triangle, the greatest side is opposite to the right angle.

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* This principle, though already demonstrated in the first section, is repeated here, in order to complete what is said on the equality of triangles.

8. When a triangle contains two equal angles, it also has two equal sides, and the triangle is isosceles.

9. If the *three* angles in a triangle are equal to each other, the sides are also equal, and the triangle is equilateral.

10. Any one side of a triangle is smaller than the sum of the two other sides.

11. If from a point within a triangle, two lines are drawn to the two extremities of one of the sides, the angle made by those lines is always greater than the angle of the triangle which is opposite to that side; but the sum of the two lines, which make the interior angle, is *smaller* than the sum of the two sides which include the angle of the triangle.

12. If from a point without a straight line, a perpendicular is let fall upon that line, and, at the same time, other lines are drawn obliquely to different points in the same straight line, the perpendicular is shorter than any of the oblique lines, and is therefore the shortest line that can be drawn from that point to the straight line.

13. The distance of a point from a straight line is measured by the length of the perpendicular, let fall from that point upon the straight line.

14. Of several oblique lines drawn from a point without a straight line, to different points in that straight line, that one is the shortest, which is nearest the perpendicular, and that one is the greatest, which is farthest from the perpendicular.

15. If a perpendicular is drawn to a straight line, then two oblique lines drawn from two points in the straight line, on each side of the perpendicular, and at equal distances from it, to any one point in that perpendicular, are equal to one another.

16. If a perpendicular is drawn to a straight line, there

is but one point in the straight line, on each side of the perpendicular, such, that a straight line drawn from it to a given point in that perpendicular, is of a given length.

17. If a perpendicular is drawn to a straight line, there is but one point in the straight line, on each side of the perpendicular, from which a line drawn to a given point in that perpendicular, makes with the straight line an angle of a required magnitude.

18. If two sides and the angle which is opposite to the greater of them, in one triangle, are equal to two sides and the angle which is opposite to the greater of them in another triangle, each to each, the two triangles coincide with each other in all their parts; that is, they are equal to each other.

19. If the hypothenuse and one side of a right-angled triangle, are equal to the hypothenuse and one side of another right-angled triangle, each to each, the two right-angled triangles are equal.

20. If in two triangles two sides of the one are equal to two sides of the other, each to each, but the angle included by the two sides in one triangle, is greater than the angle included by them in the other, the side opposite to the greater angle in the one triangle, is greater than the side opposite to the smaller angle in the other triangle.

21. Every parallelogram is, by a diagonal, divided into two equal triangles.

22. The opposite sides of a parallelogram are equal to each other.

23. The opposite angles in a parallelogram are equal to each other.

24. By *one* angle of a parallelogram all *four* angles are determined.

25. A quadrilateral, in which the opposite sides are respectively equal, is a parallelogram.

26. A quadrilateral, in which two sides are equal and parallel, is a parallelogram.

27. If from one of the vertices of a rectilinear figure, diagonals are drawn to all the other vertices, the figure is divided into as many triangles as it has sides less two.

28. The sum of all the angles in a rectilinear figure, is equal to as many times two right angles as the figure has sides less two.

RECAPITULATION OF THE TRUTHS CONTAINED IN PART II.

1. *On Proportions.*

Ques. 1. How is a geometrical ratio determined?

Q. 2. What is the ratio of a line 3 inches in length, to a line of 12 inches? What, the ratio of a line 2 inches in length, to one of 10 inches, &c.?

Q. 3. When two geometrical ratios are equal to one another, what do they form?

Q. 4. What is a geometrical proportion?

Q. 5. What signs are used to express a geometrical proportion?

Q. 6. What sign is put between the two terms of a ratio?

Q. 7. What sign is put between the two ratios of a proportion?

Q. 8. What are the first and fourth terms of a geometrical proportion called?

Q. 9. What are the second and third terms of a geometrical proportion called?

Q. 10. What are the most remarkable properties of geometrical proportions?

Ans. a. In every geometrical proportion the two ratios may be inverted.

b. In every geometrical proportion the order of the means or extremes may be inverted.

c. If two geometrical proportions have a ratio common, the two remaining ratios make again a proportion.

d. If you have several geometrical proportions, of which the second has a ratio common with the first, the third a ratio common with the second, the fourth a ratio common with the third, &c., the sum of all the first terms will be in the same ratio to the sum of all the second terms, as the sum of all the third terms is to the sum of all the fourth terms; that is, the sums make again a proportion.

e. The second term of a proportion being added once, or any number of times, to the first term, and the fourth term the same number of times to the third term, they will still be in proportion; and in the same manner can the first term be added a number of times to the second term, and the third the same number of times to the fourth term, without destroying the proportion.

f. The second term may also be once, or any number of times, subtracted from the first term, and the fourth from the third term, without destroying the proportion; or the first term may also be subtracted from the second, and the third from the fourth—and the result will still be a geometrical proportion.

g. If all the terms of a geometrical proportion are multiplied or divided by the same number, the proportion remains *unaltered.*

h. From three terms of a geometrical proportion the fourth term can be found.

i. If four lines are together in a geometrical propor-

tion, their lengths, expressed in numbers of rods, feet, inches, &c., are in the same proportion.

k. In every geometrical proportion, the product of the two mean terms is equal to that of the two extremes.

l. When the two mean terms of a geometrical proportion are equal to each other, either of them is called a mean proportional between the two extremes.

QUESTIONS ON SIMILARITY OF TRIANGLES.

Ques. What other principles do you recollect in the second part of the second section?

Ans. 1. If one side of a triangle is divided into any number of equal parts, and then, from the points of division, lines are drawn parallel to one of the two other sides, the side opposite to the one that has been divided will, by these parallels, be divided into as many equal parts as the first side.

2. If, in a triangle, a line is drawn parallel to one of the sides, that parallel divides the two other sides into such parts as are in proportion to each other and to the whole of the two sides themselves; and the reverse of this principle is also true; namely, a line must be parallel to one of the sides of a triangle, if it divides the two other sides proportionally.

3. If, in a triangle, a line is drawn parallel to one of the sides, the triangle which is cut off by it is similar to the whole triangle.

4. If the three angles in one triangle are equal to the three angles in another triangle, each to each, the two triangles are similar to one another; and the same is the case if only *two* angles in one triangle are equal to two angles in another, each to each.

5. If an angle in one triangle is equal to an angle in another, and the two sides which include that angle in the one triangle are in proportion to the two sides which include the equal angle in the other, these two triangles are similar to each other.

6. If the three sides of one triangle are in proportion to the three sides of another, the two triangles are similar to each other.*

7. If, in a right-angled triangle, a perpendicular is let fall from the vertex of the right angle upon the hypothenuse, that perpendicular divides the whole of the triangle into two parts, which are similar to the whole triangle, and to each other.

8. The perpendicular let fall from the vertex of a right-angled triangle upon the hypothenuse, is a mean proportional between the parts into which it divides the hypothenuse.

9. In every right-angled triangle, each of the sides which include the right angle is a mean proportional between the hypothenuse and that part of it, which lies between the extremity of that side and the foot of the perpendicular let fall from the vertex of the right angle upon the hypothenuse.

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* The teacher will do well to let the pupil repeat the different cases where two triangles are similar to each other. (Page 73.)

SECTION III.

OF THE MEASUREMENT OF SURFACES.

Preliminary Remarks. We determine the length of a line, by finding how many times another line, which we take for the *measure*, is contained in it. The line which we take for the measure is chosen at pleasure; it may be an inch, a foot, a fathom, a mile, &c. If we have a line upon which we can take the length of an inch 3 times, we say that line measures 3 inches, or is 3 inches long. In like manner, if we have a line upon which we can take the length of a fathom 3 times, we call that line 3 fathoms, &c. To find out which of two lines is the greater, we must *measure* them. If we take an inch for our measure, that line is the greater, which contains the greater number of inches. If we take a foot for our measure, that line is the greater, which contains the greater number of feet, &c.

To measure the extension of a surface, we make use of another surface, commonly a square (□), and see how many times it can be applied to it; or, in other words, how many of those squares it takes to cover the whole surface. The length of the square side is arbitrary. If it is an inch, the square of it is called a *square inch;* if it is a foot, a *square foot;* if it is a mile, a *square mile*, &c. The extension of a surface, expressed in numbers of square miles, rods feet, inches, &c., is called its *area*.

Remark 2. If we take one of the sides of a triangle for the *basis,* the perpendicular let fall from the vertex of the *opposite angle,* upon that side, is called the *altitude* or *height* of the triangle.

If, in the triangle ABC, (Fig. I.) for instance, we call AC the basis, the perpendicular BD will be its height. If the perpendicular BD should fall without the triangle ABC (as in Fig. II.),

Fig. I.

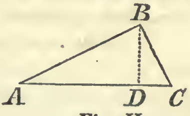
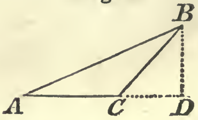
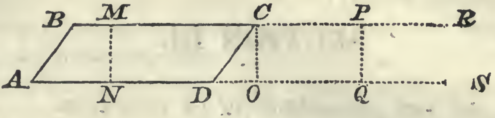


Fig. II.



we need only extend the basis, and then let fall the perpendicular BD upon its farther extension CD.



If, in a parallelogram, ABCD, we take AD for the basis, any perpendicular, MN, CO, PQ, &c., let fall from the opposite side BC, or its farther extension CR, upon that basis, or its farther extension DS, measures the height of the parallelogram. For in a parallelogram the opposite sides are parallel to each other (see Definitions), and all the perpendiculars, let fall from one of two parallel lines to the other, are equal (Query 11, Sect. I.). What in this respect holds of a parallelogram is applied also to a square, a rhombus, and a rectangle; for these three figures are only modifications of a parallelogram. (See Definitions.)

As in every rectangle, ABCD, the adjacent sides, AB, BD, are *perpendicular* to each other, it is evident that if AB is taken for the basis, the side BD itself is the height of the rectangle.

Remark 3. We call two geometrical figures equal* to one another, when they have equal areas (see preliminary remark to Sect. II.). Thus a *triangle* is said to be equal to a *rectangle* when it contains the same number of square miles, rods, feet, inches, &c., as that rectangle.

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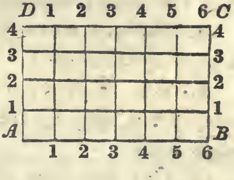
* The term *equivalent* would undoubtedly be better; but as there is no generally adopted sign in mathematics to express that two things are equivalent without being exactly the same, we are [illegible] the term *equal*.

QUERY I.

If the basis, AB, of a rectangle, ABCD, measures 6 inches, and the height, the side BC, 4 inches, how many square inches are there in the rectangle?

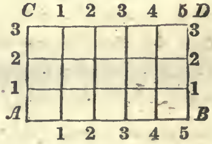


A. Twenty-four.

Q. How can you prove this?

A. If a rectangle is four inches high, I can divide it, like the rectangle ABCD (see the figure), into four rectangles, each of which is *one* inch high, and has its basis equal to the basis of the whole rectangle. And as the basis, AB, of the rectangle measures 6 inches, by raising upon it, at the distance of an inch from each other, the perpendiculars 1, 2, 3, 4, 5, each of the four rectangles will be divided into 6 square inches; and therefore the whole rectangle ABCD into 24 square inches.

Q. How many square inches are there in a rectangle, whose basis is 5, and height 3 inches?



A. Fifteen. Because in this case I can divide the rectangle into 3 rectangles of 5 square inches each.

Q. Supposing the measurements of the first rectangle were given in feet, in rods, or in miles, instead of inches, how many square feet, rods, or miles would there be in the rectangle?

A. If its measurements were given in feet, it would contain 24 square feet; if they were given in rods, it would contain 24 square rods, &c.; for in these cases I need only imagine the lines, 1, 2, 3, 4, &c., to be drawn a foot, a rod, &c., apart; the *number* of divisions will

remain the same; nothing but their size will be altered. And the same reasoning applies to the second rectangle.*

Q. Can you now give a general rule for finding the area of a rectangle?

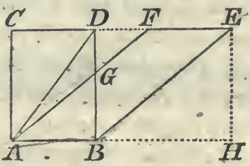
A. Yes. Multiply the length of the basis given in rods, feet, inches, &c., by the height expressed in units of the same kind.

Q. Can you now tell me how to find the area of a square?

A. The area of a square is found by multiplying one of its sides by itself. For a square is a rectangle whose sides are all equal (see Definitions); and the area of a rectangle is found by multiplying the basis by an adjacent side.

QUERY II.

If a parallelogram ABEF stands on the same basis, AB, as a rectangle, ABCD, and has its height equal to the height of that rectangle, what relation do the areas of these two figures bear to each other?



A. The area of the parallelogram ABEF is equal to the area of the rectangle ABCD; therefore I can say that the parallelogram ABEF is equal to the rectangle ABCD (see remark 3d, Introd. to Sect. III.).

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* The teacher may also give his pupils a rectangle whose measurements are both given in fractions; for instance, a rectangle of 3½ inches in length and 2¼ inches high, and then show by the figure that this rectangle measures 6 square inches, 2 half square inches, ¾ and ⅛ of a square inch; in the whole 7⅞ square inches, which is the answer to the multiplication of 3½ by 2¼.

¼	¼	¼	⅛
1	1	1	½
1	1	1	½

Q. How can you prove it?

A. The right-angled triangle ACF has the hypothenuse AF and the side AC, equal to the hypothenuse BE and the side BD, in the right-angled triangle BDE, each to each (AF and BE, AC and DB, being opposite sides of the parallelogram ABEF, and the rectangle ABCD, respectively); therefore these two triangles are equal (page 47); and by taking from each of the two equal triangles ACF, BDE, the part DGF common to both, the remainders, AGDC, BGFE, are also equal (Truth IV.); and then, by adding again to each of the equal remainders the same triangle ABG, the sums, that is, the rectangle ABCD and the parallelogram ABEF are equal to one another. (Truth III.)

Q. What important truths can you infer from the one you have just demonstrated?

A. 1st. *All parallelograms, which have equal bases and heights, are equal to one another; for each of them is equal to a rectangle upon the same basis, and of the same height.* (Truth I.)

2dly. *Parallelograms upon equal bases, and between the same parallels, are equal to one another; for if they are between the same parallels, their heights must be equal.* (Query 11, Sect. I.)

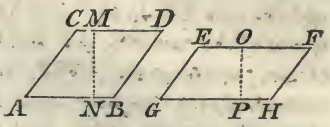
3dly. *The area of a parallelogram is found by multiplying the basis, given in rods, feet, inches, &c., by the height, expressed in units of the same kind. Because the area of the rectangle upon the same basis and of the same height to which it is equal, is found in the same manner.*

4thly. *The area of a rhombus or lozenge is found like that of a parallelogram, a lozenge being only a peculiar kind of parallelogram.*

5thly. *The areas of parallelograms are to each other, as the products obtained by multiplying the length of the*

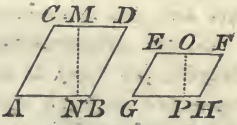
bases of the parallelograms by their heights; because these products are *the areas of the parallelograms.*

The parallelogram ABCD, for instance, is to the parallelogram GHEF, as the product of the basis AB, by the height MN, is to the product of the basis GH, by the height OP; because AB multiplied by MN is the area of the parallelogram ABCD, and GH multiplied by OP is the area of the parallelogram GHEF. This proportion may be expressed thus:



Parallelogram ABCD : parallelogram GHEF = AB × MN : GH × OP.

6thly. *Rectangles or parallelograms, which have equal bases, are to each other as their heights.*



For if, in the above proportion, the basis AB is equal to the basis GH, you can write AB instead of GH, and thereby change it into

Parallelograms ABCD : parallelograms GHEF = AB × MN : AB × OP;

that is, the parallelogram ABCD is to the parallelogram GHEF, as AB times the height MN is to AB times the height OP; or, which is the same, as the height MN alone is to the height OP alone; which is written thus:

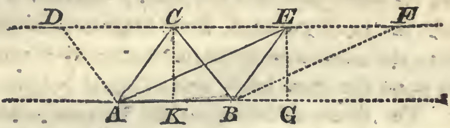
Parallelogram ABCD : parallelogram GHEF = MN : OP.

7thly. *In precisely the same manner it may be proved, that if the heights MN and OP are equal, the parallelograms ABCD, GHEF, are to each other as their bases; which may be expressed thus:*

Parallelogram ABCD : parallelogram GHEF = AB : GH.

QUERY III.

If two triangles, ABC, ABE, stand on the same basis AB, and have equal heights CK, EG, what relation do the areas of these triangles bear to each other?



A. *The areas of these triangles are equal.*

Q. How can you prove it?

A. Draw the line AD parallel to BC; BF parallel to AE; and through the two vertices C and E, the line DF parallel to AG (which is possible since the heights CK and EG are equal). The area of the parallelogram ABCD will be equal to the area of the parallelogram ABEF (Query 2, Sect. III.); and as the triangle ABC is half of the parallelogram ABCD (Query 12, Sect. II.), and the triangle ABE half of the parallelogram ABEF, the areas of these two triangles are also equal to one another; for if the *wholes* are equal, the *halves* must be equal; and in the same way it may be proved that triangles, which have *equal* bases and heights, are equal to one another.

Q. *What consequences follow from the principle just advanced?*

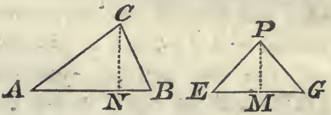
A. 1st. *Every triangle is half of a parallelogram upon equal basis and of the same height.* (This is evident from looking at the figure, and from Query 12, Sect. II.)

2d. *The* area *of a triangle is half of the area of a parallelogram upon the same basis and of the same height. Thus the area of a triangle is found by multiplying its*

basis by its height, and dividing the product by 2;* for the area of a parallelogram is equal to the *whole* product of the basis by the height.†

3d. *The areas of triangles upon the same basis and between the same parallels are equal;* because if they are between the same parallels, their heights are equal; and we have the same case as in the last query; namely, triangles upon the same basis, and of equal heights.

4th. *The areas of triangles are to each other as the products of their bases by their heights:* for the *halves* of these products being the *areas* of the triangles, the *whole* products must be in the same ratio. Thus the area of the triangle ABC is to the area of the triangle EGP, as the basis AB multiplied by the height CN, is



to the basis EG, multiplied by the height PM; which may be expressed thus:

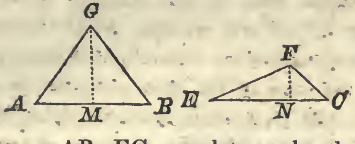
Triangle ABC : triangle EGP $=$ AB $\times$ CN : EG $\times$ PM.

5th. *The areas of triangles upon equal bases are to each other as the heights of the triangles;* because the areas of parallelograms upon the same bases and of the same heights, are to each other in the ratio of the heights; and their halves (the areas of the triangles) must be in

* Instead of multiplying the basis by the whole height and dividing the product by 2, you may multiply the basis by half the height, or the height by half the basis.

† If the basis of a triangle is 8 feet and the height 4 feet, the area of the triangle is equal to 4 times 8, divided by 2; that is, 16 square feet; whereas the rectangle upon 8 feet basis and 4 feet high, measures 32 square feet, which is double the area of the triangle.

the same ratio.* Thus if the two triangles ABG, ECF,

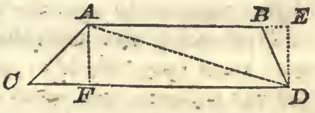


have their bases AB, EC, equal to each other, we have the proportion:

Triangle ABG : triangle ECF = CM : FN.

6th. *The areas of triangles, which have equal heights, are to each other as the bases of the triangles.* This truth follows like the preceding one from the same principle established with regard to parallelograms, of which the triangles are the halves. (Page 88, 7thly.)

QUERY IV.



How do you find the area of a trapezoid?

A. By multiplying the sum of the two parallel sides by their distance, and dividing the product by 2.

Q. How can you prove this?

A. By drawing the diagonal AD, the trapezoid ABCD, will be divided into the two triangles ACD and ABD. The area of the triangle ACD is found by multiplying its basis, CD, by its height AF, and dividing the product by 2. (Page 89, 2d.) In the same manner we find the area of the triangle ABD, by multiplying its basis, AB,

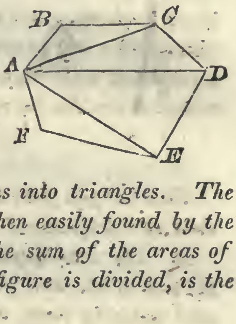
* This principle and the following one might have been established immediately from the proportion:

Triangle ABC : triangle EGP = AB × CN : EG × PM,

in precisely the same manner, as it has been proved for parallelograms. (Page 88, 6thly.)

by its height DE, and dividing the product by 2; and as the height, DE, of the triangle ABD, is equal to the height AF, of the triangle ACD (because DE and AF are perpendiculars between the same parallels), we can find the area of the two triangles, or of the whole trapezoid, ABCD, at once, by multiplying the *sum* of the two parallel lines AB, CD, by their distance AF, and dividing the product by 2.*

QUERY V.



How do you find the area of a polygon ABCDEF, or, in general, of any other rectilinear figure?

A. By dividing it by means of diagonals (as in the figure before you), or by any other means into triangles. The area of each of these triangles is then easily found by the rule given (page 89, 2d.); *and the sum of the areas of all the triangles, into which the figure is divided, is the area of it.*

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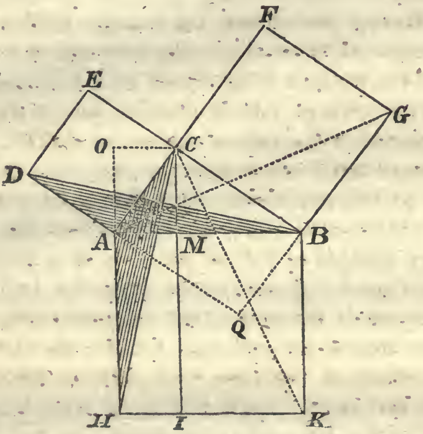
* If you multiply two numbers successively by the same number, and then add the products together, the answer will be the same as the *sum* of the two numbers at once multiplied by that number. Multiply each of the numbers, 6 and 5, for instance, by 4, and then add the products, 24 and 20, together, you will have 44; and adding, in the first place, 6 to 5, and then multiplying the sum, 11, by 4, you will again have 44.

Instead of multiplying the *sum* of the two parallel sides by their distance, and then dividing the product by 2, you may multiply, at once, *half* the sum of the two parallel sides by their distance; or the *sum* of the two parallel sides by *half* their distance.

QUERY VI.

If, upon each of the three sides AB, AC, BC, of a right-angled triangle ABC, you construct a square, what relation do the squares constructed upon the sides AC, BC, bear to the square constructed upon the hypothenuse, AB?



A. The square ABHK, constructed upon the hypothenuse AB, equals, in area, the two squares ACDE, BCGF, constructed upon the two sides AC, BC.

Q. How can you prove it by this diagram, in which the perpendicular CM, is let fall from the vertex C, of the right-angled triangle ABC, upon the hypothenuse AB, and extended until, in I, it meets the side HK, opposite to the hypothenuse; and DB and CH are joined?

A. In the first place, I should remark that the two sides AB, AD, of the triangle ABD, are equal to the two sides AH, AC, of the triangle ACH, each to each (AH

and AB, being sides of the same square, ABHK; and, AC and AD, being sides of the square ACDE); and that the angle DAB, included by the sides AD, AB, is also equal to the angle CAH, included by the two sides AC, AH (for each of these angles is formed by the angle CAB being added to the right angle of a square); therefore these two triangles are equal to each other. (Query 1, Sect. II.)

Q. Having proved that the triangle ABD is equal to the triangle ACH, what can you infer from it?

A. That the area of the square ACDE, is equal to the area of the rectangle AHIM. For the area of the triangle ABD, is half of the area of the square ACDE; because the triangle ABD stands upon the same basis AD, as the square ACDE, and has its height BQ, equal to the height AC of that square; and it has been proved that the area of every triangle is half of the area of a rectangle or square of equal basis and height. (Page 89, 1st.) For the same reason is the area of the triangle ACH, equal to half the area of the rectangle AHIM; for the triangle ACH, stands on the same basis AH, as the rectangle AHIM, and has its height CO, equal to the height AM, of that rectangle; and as the *halves*, the two triangles ABD and ACH, are equal to each other, the *wholes*, the squares ADEC and AHIM, must also be equal to each other. In precisely the same manner I can prove from the equality of the two triangles ABG and BCK, that the square BCFG is equal to the rectangle MBIK; and because the area of the square ADEC, is equal to the area of the rectangle AHIM, and the area of the square BCFG is equal to the area of the rectangle MBIK; therefore the *sum* of the areas of the two rectangles, AHIM and MBIK, that is, the area of the *square upon the hypothenuse AB, is equal to the sum of the*

areas of the squares constructed upon the two sides AC, BC.

Remark. For the discovery of this principle, we are indebted to Pythagoras, a famous Greek mathematician. It is a very important one, and teaches how to find one of the sides of a right-angled triangle when the two others are given. If, for instance, the two sides AC, BC, of the right-angled triangle ABC, were known to measure, one 5, the other 6 inches, the sum of their squares 25 (5 times 5), and 36 (6 times 6), equal to 61, would be the area of the *square* of the hypothenuse; and the square *root* of that number would be the hypothenuse AB, itself. If the hypothenuse and one of the sides are given, you need only *subtract* the square of the side from the square of the hypothenuse, and then the square root of the remainder is the other side. If, for instance, the hypothenuse of a right-angled triangle were 10 feet, and one of the sides 6 feet; the square of the hypothenuse would be 10 times 10, or 100, and the square of 6, which is 36, subtracted from 100, leaves 64, which would be the *square* of the side to be found; and taking the square root of it, which is 8 (because 8 times 8 are 64), you will have the side itself.*

QUERY VII.

It has been proved (page 90, 4th), *that the areas of any two triangles are to each other as their bases multiplied by their heights; can you now find out the relation which the bases and heights of* similar *triangles bear to each other?*

A. In similar triangles the bases are in proportion to the heights.

Q. How can you prove this?

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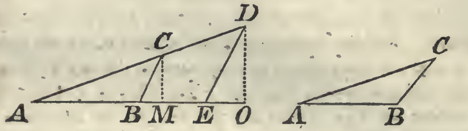
* We shall hereafter give the geometrical solutions of these problems.

A. Let there be any two similar triangles ABC, AED. Place the smaller one ABC, upon the larger AED, in



such a way that the angle at A falls upon the angle at A, and from the vertices, C and D, let fall the perpendiculars CM, DO, upon AO. Then the two triangles BCM, EDO, are both right-angled, and the angle CBM is equal to the angle DEO (because in the two similar triangles ABC, AED, the angles ABC and AED, are equal to each other, and CBM and DEO, make with them, respectively, two right angles); therefore the third angle BCM in the triangle BCM, is also equal to the third angle EDO, in the triangle EDO, and the two triangles BCM, EDO, are similar. (Page 73, 1st.) But in similar triangles the sides opposite to the equal angles are proportional, consequently we have

$$CM : DO = CB : DE;$$

and in the similar triangles ABC, ADE,

$$AB : AE = CB : DE.$$

These two proportions have the second ratio common; therefore the two first ratios must again make a proportion (Theory of Proportions, Principle 3d.), namely:

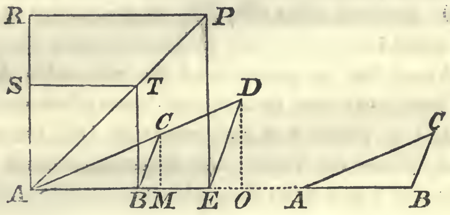
$$AB : AE = CM : DO.$$

This proportion expresses that the bases AB of the smaller triangle ABC, is to the bases AE of the larger triangle AED, as the height CM, of the first triangle ABC, is to the height DO, of the triangle AED.

QUERY VIII.

From what you have learned in the preceding query, can you determine the proportion which the areas *of similar triangles bear to each other?*

A. The areas of similar triangles are to each other as the squares upon the corresponding sides.



Q. How can you prove this, for instance, of the two similar triangles ABC, ABD?

A. Let us place the smaller triangle ABC upon the larger AED, as in the last query; and upon AB and AE, construct the squares ABST, AERP. Then the triangles ABC, AED, have *the same bases* AB, AE, as the triangles ABT, AEP, and *their heights* CM, DO, *are in proportion* to the heights TB, PE, of the triangles ABT, AEP (TB and PE being respectively equal to AB, AE, which, in the last query, are proved to be proportional to CM and DO); therefore the *areas* of the triangles ABC, AED, are in proportion to the areas of the triangles ABT, AEP. (Page 90, 5thly.) But if the two triangles ABC, AED, are in proportion to the two triangles ABT, AEP, which are the *halves* of squares ABST, AERP, they must also be in proportion to the squares themselves; which may be expressed thus:

Triangle ABC : triangle AED $=$ AB $\times$ AB : AE $\times$ AE, and is read :

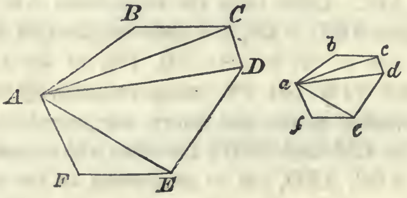
The area of the triangle AED is as many times greater than the area of the triangle ABC, as the area of the square upon the side AE, is greater than the area of the square upon the corresponding side AB.

Q. Can you prove that the same ratio exists also between the squares upon the sides AC and AD, and also between the two sides CB, DE, of the similar triangles ABC, AED ?

A. Yes. For to prove it of the two sides AC and AD, I need only take *them* for the bases of the two triangles; and to prove it of the sides CB, DE, I must take CB and DE for the bases; the reasoning would be the same as that I just went through.

QUERY IX.

From the ratio which you have proved to exist between the areas of similar triangles, *can you now find out the ratio which exists between the areas of similar* polygons ? (See Definitions.)



A. Yes. *The areas of similar polygons are to each other, as the areas of the squares constructed upon the corresponding sides.* The areas of the two similar polygons ABCDEF, *abcdef*, for instance, are to each other as the

areas of the squares constructed upon the sides AB, *ab*, or as the areas of the squares upon the sides BC, *bc*, &c. For, by drawing in the polygon ABCDEF, the diagonals AC, AD, AE, and in the polygon *abcdef*, the corresponding diagonals, *ac*, *ad*, *ae*, the triangle ACB, is similar to the triangle *abc*, the triangle ACD, to the triangle *acd*, &c.; because, if the whole polygons ABCDEF, *abcdef*, are similar, their similarly disposed parts must also be similar; and the same proportion which exists between their parts, must necessarily exist also between the whole polygons; consequently, as the areas of the triangles ABC, *abc*, ACD, *acd*, &c., are in the ratio of the areas of the squares constructed upon their corresponding sides, the whole polygons must be in the same ratio, which may be expressed thus:

$$\text{Polygon ABCDEF} : \text{polygon } abcdef = AB \times AB : ab \times ab.$$

* * *

RECAPITULATION OF THE TRUTHS IN THE THIRD SECTION.

Ques. 1. How do you determine the length of a line?

2. How do you find out which of two lines is the greater?

3. How can you measure a surface?

4. What do you call the area of a surface?

5. If you take one of the sides of a triangle for the basis, how do you determine the height of the triangle?

6. How is the height of a parallelogram determined? How that of a rectangle? A rhombus? A square?

7. When do you call a triangle equal to a square? to a parallelogram? to a rectangle, &c.?

8. When can you call two geometrical figures equal to one another, though these figures do not coincide with each other?

9. Can you repeat the different principles respecting the areas of geometrical figures, which you have learned in this section?

Ans. 1. The area of a rectangle is found by multiplying its basis, given in miles, rods, feet, inches, &c., by its height expressed in units of the same kind.

2. The area of a square is found by multiplying one of its sides by itself.

3. If a parallelogram stands on the same basis as a rectangle, and has its height equal to the height of that rectangle, the area of the parallelogram is equal to the area of the rectangle?

4. The areas of all parallelograms, which have equal bases and heights, are equal to one another.

5. Parallelograms upon equal bases, and between the same parallels, are equal to one another.

6. The area of a parallelogram is found by multiplying the basis given in rods, feet, inches, &c., by the height, expressed in units of the same kind.

7. The area of a rhombus or lozenge is found like that of a parallelogram.

8. The areas of parallelograms are to each other, as the products obtained by multiplying the bases of the parallelograms by their heights.

9. Rectangles, or parallelograms which have equal bases, are to each other as their heights.

10. Rectangles, or parallelograms which have equal heights, are to each other as their bases.

11. If two triangles stand on the same basis, and have equal heights, their areas are equal to one another.

12. Every triangle is half of a parallelogram upon equal basis and of the same height.

13. The area of a triangle is half of the area of a parallelogram upon equal basis and of the same height; and, therefore, the area of a triangle is found by multiplying the length of its basis by its height, and dividing the product by 2.

14. The areas of triangles upon the same basis, and between the same parallels, are equal.

15. The areas of triangles are to each other, as the products of their bases by their heights.

16. The areas of triangles upon equal bases are to each other, as the heights of the triangles.

17. The areas of triangles, which have equal heights, are to each other, as their bases.

18. The area of a trapezoid is found by multiplying half the sum of the two parallel sides, by their distance.

19. The area of any rectilinear figure, terminated by any number of sides, is found by dividing that figure, either by diagonals or by any other means, into triangles, and then adding the areas of these triangles together.

20. If, upon each of the three sides of a right-angled triangle, a square is constructed, the square upon the hypothenuse equals, in area, the two squares constructed upon the two sides, which include the right angle.

21. The bases of similar triangles are to each other, as the heights of the triangles.

22. The areas of similar triangles are to each other, as the areas of the squares upon the corresponding sides.

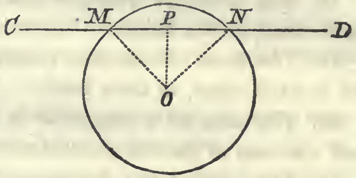
23. The areas of similar polygons are to each other, as the squares constructed upon the corresponding sides.

SECTION IV.

OF THE PROPERTIES OF THE CIRCLE.*

QUERY I.

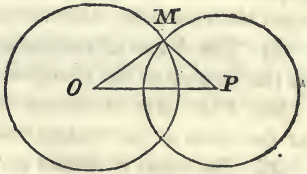
In how many points can a straight line, CD, meet the circumference of a circle?



A. In two points, M, N, only. For, letting fall, from the centre of the circle, the perpendicular OP upon the straight line CD, there is but *one* point in the line CD, on each side of the perpendicular, such, that a line, drawn from it to the point O of the perpendicular, has the length of the radius ON. (Page 46, 6thly.)

QUERY II.

In what cases do the circumferences of two circles cut each other?



A. When the distance, OP, between their centres, O and P, is less than the sum of their radii, OM, PM.

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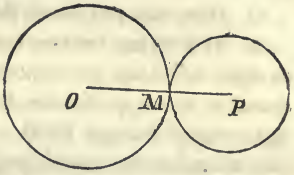
* Before entering on this section, the teacher ought to recapitulate with his pupils the definitions of a circle, of an arc, of a chord, a segment, &c.

QUERY III.

When do two circles touch each other exteriorly?

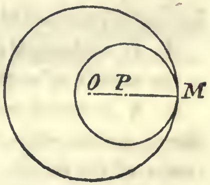
A. When the distance, OP, between their centres, O and P, is equal to the sum of their radii OM, PM.



QUERY IV.

When do two circles touch each other interiorly?

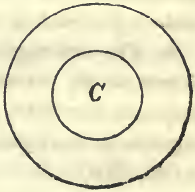
A. When the distance, OP, between their centres, O and P, is equal to the difference between their radii, OM and PM.



QUERY V.

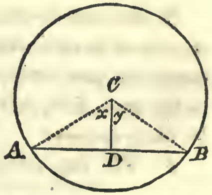
When are the circumferences of two circles parallel to each other?

A. When they are concentric, that is, when they are described from the same point, C, as the centre.



QUERY VI.

If, from the centre, C, of a circle, a perpendicular, CD, is let fall upon a chord, AB, in that circle, what relation do the two parts, AD, BD, into which the chord AB is divided, bear to each other?



A. The two parts AD, BD, are equal to each other; that is, the chord AB is bisected in the point D.

Q. How can you prove this?

A. By drawing the two radii AC, BC, the right-angled triangle ACD has the hypothenuse AC, and the side CD, equal to the hypothenuse BC, and the side CD, in the right-angled triangle BCD, each to each; therefore these two triangles are equal (page 47); and the side AD, in the triangle ACD, is equal to the side BD in the equal triangle BCD.

Q. What other truths can you infer from the one you have just established?

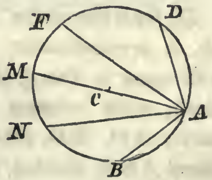
A. 1. *A straight line, drawn from the centre of a circle to the middle of a chord, is perpendicular to that chord.*

2. *A perpendicular, drawn through the middle of a chord, passes, when sufficiently far extended, through the centre of the circle.*

3. *Two perpendiculars, each drawn through the middle of a chord in the same circle, intersect each other at the centre; for each of them must go through the centre.*

4. *The two angles,* x *and* y, *which the radii AC, BC, drawn to the extremities of the chord AB, make with the perpendicular CD, are equal to one another;* for they are opposite to the equal sides AD, BD, in the equal triangles ADC, BDC.

QUERY VII.



If the two chords, AD, AB, are equal to each other, what remark can you make with regard to the arcs AD, AB, subtended by these chords?*

A. The two arcs, AB, AD,

* The arcs AD, AB, standing on the chords AB, AD, are said to be subtended by these chords.

subtended by the equal chords, AB, AD, are equal to one another.

Q. Why?

A. This follows from the perfect *uniformity* with which a circle is constructed. For, if the chord AB is placed upon its equal, the chord AD, the arcs, AB and AD, must coincide with each other; because every point in both these arcs is at the same distance from the centre, C, of the circle.

Remark. It is to be observed that each chord subtends *two* arcs, one of which is *smaller*, and the other *greater* than the semi-circumference, both together completing the *whole* circumference. In speaking of an arc, subtended by a chord, we always mean that which is *smaller* than the semi-circumference.

Q. What other truths can you infer from the one you have just proved?

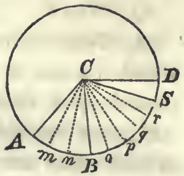
A. 1. *That equal arcs stand on equal chords;* for, by placing one of the equal arcs AB, AD, upon the other, the *beginning* and *end* of the two chords AB, AD, and therefore the *whole* chords themselves, coincide with each other.

2. *The greater arc stands on the greater chord, and the greater chord subtends the greater arc.* The chord AF, for instance, is greater than the chord AD; because the arc AF, belonging to the greater chord AF, is greater than the arc AD, belonging to the smaller chord AD.

3. *Among all the chords, AD, AF, AM, AN, AB, &c., which can be drawn in a circle, the diameter AM is the greatest;* because the greatest arc, the semi-circumference, stands on it.

Remark. All that has been said of chords and arcs in the *same* circle, holds true also of chords and arcs in *equal* circles.

QUERY VIII.



What relation do you discover between the angles ACB, BCD, at the centre, C, of a circle, and the arcs AB, BD, intercepted between their legs?

A. The angles ACB, BCD, at the centre, are to each other in the same ratio as the arcs AB, BD, of the circumference.

Q. How can you show this?

A. I divide the whole of the arc AD successively into smaller and smaller parts, until one of the points of division shall have fallen upon B. Then, it is evident that by drawing to the points of division the radii C*m*, C*n*, C*o*, &c., the *angles* ACB and BCD are divided into as many equal parts as the *arcs* AB, BD (for the sectors AC*m*, *m*C*n*, *n*CB, &c., will all coincide with each other, when they are placed upon one another); and therefore the same ratio which exists between the arcs AB, BD, exists also between the angles ACB, BCD. In our figure, we have the ratio of the arc AB to the arc BD as 3 to 6; and the same ratio (as 3 to 6) exists also between the angles ACB and BCD at the centre of the circle; that is, the arc BD is as many times greater than the arc AB, as the angle BCD is greater than the angle ACB (Def. of Geom. Proportions).

What inference can you draw from the truth you have just advanced?

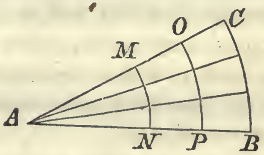
Ans 1. *If the arcs AB, BD, are equal to one another, the angles ACB, BCD, at the centre, are also equal to one another;* for they are in the same ratio as the arcs AB, BD (namely, then, in the ratio of equality).

2. *If the angles ACB, BCD, at the centre, are equal to one another, the arcs AB, BD, are also equal to one another;* because they are to each other in the same ratio as the angles at the centre.

Remark 1. It has already been stated (note to page 12), that angles are measured by arcs of circles, described with any radius between their legs. The reason is now apparent; for the arcs intercepted between their legs are in *proportion* to the angles at the centre.

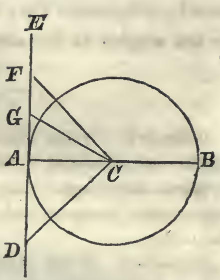
Remark 2. If the circumference of a circle is divided into 360 equal parts, called degrees; each degree again into 60 equal parts, called minutes; each minute again into 60 equal parts, called seconds, &c.; it is easy to perceive, that the magnitude of an angle does not depend upon the *length* of the arc intercepted between its legs; but merely upon the *number of degrees, minutes, seconds,* &c., it measures of the circumference of the circle of which it is a part.

Thus, if the angle BAC measures 3 degrees by the arc MN, it measures the same number of degrees by the arc OP, the same number of degrees by the arc CB, &c., although the degrees themselves vary in size.



Remark 3. As the sum of all the angles around the same point is equal to 4 right angles (page 24), the sum of all the angles around the centre of a circle is also equal to 4 right angles; therefore the circumference of a circle is the measure of 4 right angles; the semi-circumference that of 2 right angles, and the arc of a quadrant, that of one right angle. If the circumference of a circle is divided into 360 degrees, 90 of them are the measure of 1 right angle, 180 that of 2 right angles, and 360 that of 4 right angles.

QUERY IX.



If a straight line is drawn perpendicular to the extremity A, of the radius AC, in how many points will that line meet the circumference of the circle?

A. In one only (namely, the point A); and therefore the line DE is a tangent to the circle. (See Def. page 15.)

Q. But why can the line ED have no other point common with the circumference?

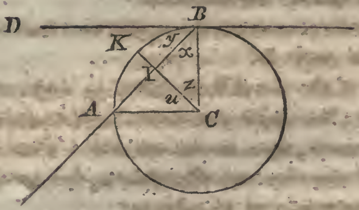
A. Because the perpendicular AC is the shortest line which can be drawn from the point C, the centre of the circle, to the straight line ED (page 44); therefore every other line, CG, CF, CD, drawn from the centre, C, to the straight line ED, will be *greater than the radius AC;* consequently every point in the line ED, except the point A itself, is *without* the circle.

Q. And what other truths can you infer from the one last established?

A. 1. A radius or diameter, drawn to the point of tangent, is perpendicular to the tangent.

2. A line drawn through the point of tangent perpendicular to the tangent, passes, when sufficiently far extended, through the centre of the circle.

QUERY X.



What relation does the angle ACB, measured by the arc AB bear to the angle y, *formed by the tangent BD and the chord AB, which subtends the arc AB?*

A. The angle ACB, at the centre of the circle, is twice as great as the angle y, *formed by the tangent BD, and the chord AB.*

Q. How can you prove this?

A. From the centre of the circle, let fall the perpendicular CI upon the chord AB, and extend it until it meets the circumference in K. Then the angles *w* and *z*, and consequently the arcs AK, KB, are equal to one another (page 104, 4thly). We have further the triangle BIC right-angled, and therefore the two angles *x* and *z*, together, equal to a right angle (page 34, 7thly); and because the tangent DB is perpendicular to the radius CB, the angles *x* and *y* are together also equal to a right angle; therefore the angle *z* is equal to the angle *y* (Truth III, page 21): and as the angle *z* is half of the angle ACB, the angle *y* (its equal) is also half of the angle ACB.

Q. And what remark can you make with regard to the arc BK?

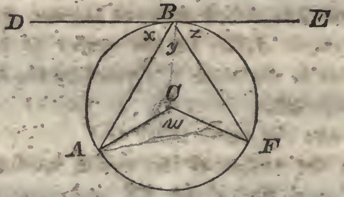
A. That the arc BK, which measures the angle z, *may be taken also for the measure of the angle* y *(its equal);*

and as the arc BK is half of the arc AB, the angle y, *made by the tangent BD and the chord AB, may likewise be measured by half the arc AB.*

Q. What do you mean by saying that half the arc AB measures the angle y?

A. That if the arc AB is given in degrees, minutes, seconds, &c., the angle y measures half as many degrees, minutes, seconds, &c., as the arc AB. Thus if the arc AB were 12 degrees and 30 minutes, the angle y would measure 6 degrees and 15 minutes.

QUERY XI.



What relation does the angle w, *formed by the two radii CA, CF, bear to the angle* y, *formed by the two chords AB, FB, if the legs of both these angles stand on the extremities of the same arc AF?*

A. The angle w, *formed by the two radii CA, CF, is twice as great as the angle* y, *formed by the two chords AB, FB.*

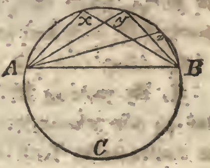
Q. How can you prove it?

A. Drawing in the point B a tangent, DE, to the circle, the angles x, y, z, being together equal to two right angles (Query 4, Sect. I.), will have for their measure half the circumference of the circle (page 107, remark 3d). Now, the angle x, formed by the tangent DB and the chord AB, is measured by half the arc AB, as has been proved in the last query; and for the same

reason is the angle z measured by half the arc BF; and therefore the remaining angle y is measured by half the arc AF; because half of the arc AF makes with half of the arcs AB and BF, half the circumference. But the angle, w, at the centre is measured by the *whole* arc AF; therefore the angle w is twice as great as the angle y.

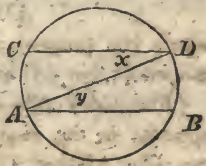
Q. What important truths can you infer from the one you have just learned?

A. That every angle made by two chords at the circumference of a circle, measures half as many degrees, minutes, seconds, &c., as the arc on the extremity of which these chords stand.



2. *The angles* x, y, z, *at the circumference, having their legs standing on the extremities of the same arc, ACB, are all equal to one another;* because each of them is measured by half the arc ACB.*

QUERY XII.



If two chords, AB, CD, in the same circle, are parallel to each other, what relation do the arcs, AC, BD, intercepted by them, on both sides of the circumference, bear to each other?

A. The arcs AC, BD, are equal to each other.

Q. How can you prove it?

A. Joining AD, the alternate angles x and y are equal to one another (Query 10, Sect. I.); therefore the arcs

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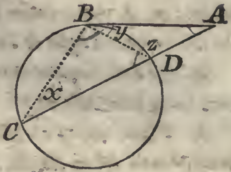
* The arc ACB is designated by three letters, in order to distinguish it from the upper arc AB.

AC and BD, measured by the halves of these angles, are also equal to one another.

QUERY XIII.



If from the same point, A, without a circle, you draw a tangent, AB, to the circle, and, at the same time, another line, AC, cutting the circle, what relation exists between the tangent AB, and the line AC, which cuts the circle?

A. *The tangent AB is a mean proportional* (Theory of Prop., page 66), *between the whole line AC and the part AD, which is without the circle.*

Q. How can you prove it?

A. By joining BD and BC, the triangle ABD is similar to the whole triangle ABC; because the angle at A is common to both triangles, and the angle y in the triangle ABD, is equal to the angle x, in the triangle ABC (both angles being measured by half the arc BD*); therefore we have the proportion

$$AD : AB = AB : AC,$$

where the tangent AB is a mean proportional between the whole line AC and the part AD without the circle.

(The sides AD and AB, in the triangle ABD, are opposite to the angles y and z in the same triangle, and the sides AB and AC, in the triangle ABC, are opposite to the angles x and CBA, which are respectively equal to the angles y and z.)

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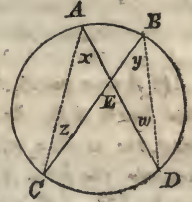
* The angle x is formed at the circumference by the two chords BC and DC, whose extremities stand on the arc BD (Query 11, Sect. IV.); and the angle y is formed by the tangent BA, and the chord BD, which subtends the arc BD. (Query 10, Sect IV.)

AC/AB = AF/AB

QUERY XIV.



If two chords, AD, BC, cut each other within the circle, what relation exists between the parts AE, ED, BE, EC, into which they mutually divide each other?

A. The two parts AE, ED, are in the inverse ratio of the two parts BE, EC; that is, we shall have the proportion

EC : EA = ED : EB.*

Q. How can you prove it?

A. Joining AC and BD, the angle w is equal to the angle z; because each of these two angles, w, z, measures half as many degrees as the arc AB; for the same reason is the angle x equal to the angle y; because each of these angles measures half as many degrees as the arc CD (Query 11, Sect. IV.); and the angles AEC, BED, are also equal to each other, being opposite angles at the vertex (Query 5, Sect. I.); therefore the three angles of the triangle AEC are equal to the three angles of the triangle BED, each to each; consequently these two triangles are similar to one another; and the sides opposite to the equal angles, in both triangles, are in the proportion

EC : EA = ED : EB.

(EC and EA are opposite to the angles x and z in the triangle AEC; and ED and EB are opposite to the an-

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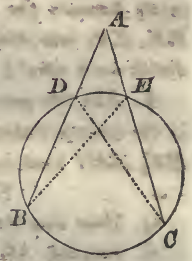
* The ratio ED to EB, is called inverse or inverted, because the two parts ED, EB, are not in *direct* proportion to the two parts EC, EA; that is, the part EC of the chord BC, is to the part EA of the chord AD, not as the other part EB of the first chord BC, is to the other part ED of the chord AD, but as the part ED of the second chord is to the part EB of the first one.

gles y and w, which are equal to the angles x and z, each to each).

QUERY XV.



If from a point, A, without a circle, two lines, AB, AC, are drawn, cutting the circle; what relation exists between the lines AB, AC, and their parts AD, AE, without the circle?

A. The whole lines, AB, AC, are to each other in the inverse ratio of their parts, AD, AE, without the circle; that is, we have the proportion,

AB : AC = AE : AD (see the note to page 113).

Q. Why is this so?

A. If you join BE and DC, the two triangles ABE and ADC are similar to each other; because two angles in the one are equal to two angles in the other, each to each (page 73, 1st); the angle at A, namely, is common to both, and the angles at B and C are equal; because they have the same measure (half the arc DE); and as in similar triangles the sides opposite to the equal angles are in proportion, we have

AB : AE = AC : AD,*

or, by changing the order of the mean terms (principle 2d of proportion),

AB : AC = AE : AD,

as above.

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* The teacher will do well to show his pupils again, that the sides AB and AE are the corresponding sides to AC and AD; because they are opposite to the equal angles in the triangles.

Remark 1. A *regular polygon* is a rectilinear figure which has all its angles and all its sides equal to one another.

Remark 2. A rectilinear figure is said to be *inscribed* in a circle, when the vertices of all the angles of that figure are at the circumference of the circle.

Remark 3. A rectilinear figure is said to be *circumscribed* about a circle, when every side of that figure is a *tangent* to the circle.

QUERY XVI.

If you divide the circumference of a circle into any number of equal parts, for instance into 6 parts, and then join the points of division by the chords AB, BC, CD, DE, EF, FA; what remark can you make respecting the rectilinear figure, ABCDEF, which will be inscribed in the circle?

A. The figure thus inscribed in the circle is a regular polygon.

Q. How can you prove this?

A. The circumference of the circle being divided into equal parts, it follows that the arcs AB, BC, CD, &c., and consequently also the chords AB, BC, CD, &c., which form the sides of the inscribed figure, are equal to one another (page 105, 1st); and as each of the angles ABC, BCD, CDE, &c., has its legs standing on the whole circumference less two of the equal arcs, into which the circumference is divided, they all measure the same number of degrees, and consequently the angles of the inscribed figure are also equal to one another;* therefore the inscribed figure ABCDEF is a *regular* polygon.

* The angle ABC, for instance, has its legs standing on the whole circumference less the two arcs AB, BC; and the angle BCD has its legs standing on the whole circumference less the two equal arcs BC, CD, &c.

Q. If in this manner you divide the circumference of a circle into 3, 4, 5, 6, *&c., equal parts, what will be the magnitude of each of the arcs AB, BC, CD, &c.?*

A. Each of the arcs AB, BC, CD, &c., will then be $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, *&c., of the whole circumference, that is,* $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, *&c., of* 360 *degrees, according as the circumference has been divided into* 3, 4, 5, 6, *&c., parts.*

Q. And what do you observe with regard to the angles x, y, z, *&c., at the centre of the circle, which the radii OA, OB, OC, &c., drawn to the points of division A, B, C, D, &c., make with each other?*

A. That these angles, x, y, z, *&c., are all equal to one another;* because they are measured by the equal arcs AB, BC, CD, &c. They will therefore measure $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., of 360 degrees, according as the circumference of the circle is divided into 3, 4, 5, &c., equal parts.

QUERY XVII.

Can you find the relation which one of the sides of a regular inscribed hexagon bears to the radius of that circle? (See the figure belonging to the last Query.)

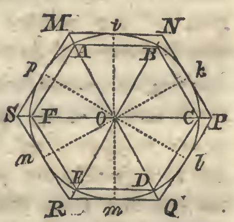
A. The side of a regular hexagon inscribed in a circle, is equal to the radius of that circle.

Q. Why?

A. Because each of the triangles ABO, BCO, CDO, &c., is in the first place isosceles, two of its sides being radii of the same circle; and as each of the angles x, y, z, &c., at the centre of the circle, measures $\frac{1}{6}$ of 360, that is, 60 degrees (last Query), it follows that the two angles at the basis of each of the isosceles triangles ABO, BCO, CDO, &c. (for instance, the two angles w and u, at the basis of the isosceles triangle ABO), measure together 120 degrees; because the sum of the three angles in every triangle is equal to two right angles, or 180 de-

grees, and 60 from 180 leave 120 degrees. Now, as the two angles at the basis of every isosceles triangle are equal to each other (Query 3, Sect. II.); each of the two angles at the basis of one of the isosceles triangles ABO, BCO, DCO, &c., will measure half of 120, that is, 60 degrees. But, each of the angles at the centre measuring also 60 degrees, the three angles in each of the triangles ABO, BCO, CDO, &c., are equal to one another; and therefore these triangles are not only isosceles, but also *equilateral*; consequently each of the sides AB, BC, CD, &c., of the hexagon is equal to the radius of the circle.

QUERY XVIII.



If, in a regular inscribed polygon, you draw from the centre of the circle the radii Oi, Ok, Ol, Om, &c., perpendicular to the chords AB, BC, CD, &c.; and at the extremities of these radii, the tangents MN, NP, PQ, &c.; what do you observe with regard to the figure MNPQRS, circumscribed about the circle?

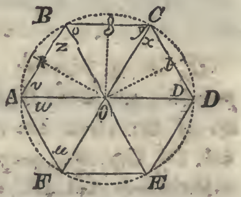
A. The figure MNPQRS, circumscribed about the circle, is a regular polygon, of the same number of sides as the inscribed polygon, ABCDEF.

Q. How can you prove this?

A. The chords AB, BC, CD, &c., are perpendicular to the same radii, to which the tangents MN, NP, PQ, &c., are perpendicular; consequently the chords AB, BC, CD, &c., are parallel to the tangents MN, NP, PQ, &c. (for two straight lines, which are both perpendicular to a third line are parallel to each other; Query 7, Sect. I.);

and therefore the triangles ABO, BCO, CDO, &c., are all similar to the triangles MNO, NPO, PQO, &c., from which they may be considered as cut off, by the lines AB, BC, CD, &c. being drawn parallel to the sides MN, NP, PQ, &c. (Query 16, Sect. II.) Now, as the triangles ABO, BCO, CDO, &c., are all equal to one another, the triangles MNO, NPO, PQO, &c., are all equal to one another. And therefore the circumscribed figure MNPQRS is a regular polygon, similar to the one inscribed in the circle.

QUERY XIX.



It has been proved (Query 16, Sect. IV.), *that a regular polygon, of any number of sides, may be* inscribed *in a circle, by dividing the circumference of the circle into as many equal parts as the polygon shall have sides, and then joining the points of division by straight lines: can you now prove the reverse, that is, that around every regular polygon, a circle can be drawn in such a manner, that all the vertices of the polygon shall be at the circumference?*

A. Yes. For I need only bisect two adjacent sides of a regular polygon; for instance, the two sides, AB, BC, of the regular polygon ABCDEF; and in the points of bisection, erect the two perpendiculars gO, kO, which will necessarily cut each other in a point, O. Then it is evident, that by drawing the lines OB, OC, OA, these three lines are equal to each other; for the line OB is equal to OC, because the two points B and C are at an equal distance from the perpendicular gO (page 45, 5thly); and for the same reason is OB also equal to OA;

because the points B and A are at an equal distance from the perpendicular kO Thus we have in the two triangles ABO, BCO, the three sides in the one, equal to the three sides in the other; therefore these two triangles are both isosceles and equal to each other.

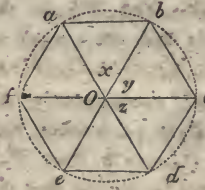
Q. But of what use is your proving that the triangle ABO is equal to the triangle BCO?

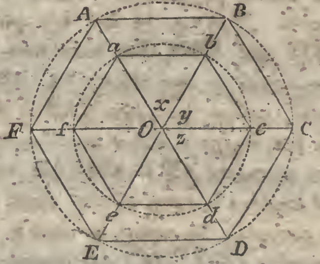
A. It shows that each of the angles in the polygon is bisected by one of the lines OA, OB, OC. For, in the first place, we have in the two equal triangles BCO, ABO, the angle o equal to the angle z; therefore the angle ABC is bisected; and the angle o is further equal to the angle y, and the angle z to the angle v; therefore the angles BCD, and FAB, are also bisected. And now I can show that, by drawing from the point O the lines OF, OE, OD, to the remaining vertices F, E, D, the whole polygon is divided into equal isosceles triangles. Taking, in the first place, the two triangles, AFO and ABO, they have two sides, OA, FA, in the one, equal to two sides, OA, AB, in the other, each to each; and as the angle FAB is bisected by the line OA, the two angles v and w are also equal; consequently, the two triangles AFO, ABO, are equal to each other, and the angle u is equal to the angle w (Query 3, Sect. II.). In precisely the same manner it may be proved that the triangles FEO, EDO, are isosceles, and equal to the triangle AFO. And as the whole polygon ABCDEF is thus divided into equal isosceles triangles, the lines OA, OB, OC, OD, OE, OF, are all equal to one another; and therefore, by describing from the point O, as a centre, with a radius OA, a circle around the polygon ABCDEF, each of the vertices A, B, C, D, E, F, will be in the circumference of the circle.

Q. What other important consequence follows from the principle you have just proved?

A. That in every regular polygon a circle may be inscribed in such a manner, that every side of the polygon is a tangent to the circle. For if, in the regular polygon ABCDEF, you describe with a radius O*g* the circumference of a circle, that circumference will touch the middle of the sides AB, BC, CD, DE, EF, FA, of the polygon ABCDEF; because the lines O*k*, O*g*, O*l*, &c. are all equal to one another, and will therefore be radii of the inscribed circle; and the sides AB, BC, CD, &c., being perpendicular to the radii O*k*, O*g*, O*l*, &c., will all be tangents to that circle. (Page 108.)

QUERY XX.





What relation do you observe to exist between two regular polygons, abcdef, *ABCDEF, of the same number of sides?*

A. They are similar to one another.

Q. How can you prove it?

A. By describing a circle around each of the regular polygons *abcdef*, ABCDEF, and drawing the radii O*a*, O*b*, O*c*, O*d*, O*e*, O*f*, OA, OB, OC, OD, OE, OF, each

of these polygons is divided into as many equal triangles as there are sides in the polygon; and as all the angles, x, y, z, &c., formed at the centre of a regular polygon, are equal to one another, I can place the centre, O, of the polygon *abcdef*, upon the centre, O, of the polygon ABCDEF, in such a manner, that the angles at the centre shall all coincide with each other; namely, so that the radius O*a* shall fall upon the radius OA, O*b* upon OB, O*c* upon OC, &c. Then, it is evident that the sides, *ab*, *bc*, *cd*, *de*, &c., of the smaller polygon, *abcdef*, are *parallel* to the sides, AB, BC, CD, DE, &c., of the greater polygon, ABCDEF; for the points *a*, *b*, *c*, *d*, *e*, *f*, and A, B, C, D, E, F, are in the circumferences of concentric circles (Query 5, Sect. IV.); therefore the triangles O*ab*, O*bc*, O*cd*, &c., in the smaller polygon, are all similar to the triangles OAB, OBC, OCD, &c., in the greater polygon (Query 16, Sect. II.); consequently the whole polygon *abcdef* is similar to the whole polygon ABCDEF.

Q. What other truths can you infer from the one you have just learned?

A. 1. *The sums of all the sides of two regular polygons of the same number of sides are to each other in the same ratio as the radii of the inscribed or circumscribed circles.* For in the two triangles ABO and *abo*, for instance, we have the proportion

$$AB : ab = AO : ao; \text{ that is,}$$

the side AB is as many times greater than the side *ab*, as the radius AO is greater than the radius *ao*; and therefore 6 or any other number of times the side AB, is as many times greater than the same number of times the side *ab*, as the radius OA is greater than the radius *oa*; that is, the sum of all the sides of the regular polygon ABCDEF, is as many times greater than the sum of all

the sides of the regular polygon *abcdef*, as the radius OA of the circle, circumscribed about the regular polygon ABCDEF, is greater than the radius, *oa*, of the circle circumscribed about the regular polygon *abcdef*. In the same manner I can prove that the sum of all the sides of the regular polygon ABCDEF, is as many times greater than the sum of all the sides of the regular polygon *abcdef*, as the radius of the circle, *inscribed* in the regular polygon ABCDEF, is greater than the radius of the circle inscribed in the regular polygon *abcdef*.

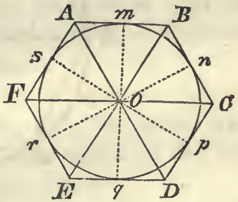
Remark. The sum of all the sides of a geometrical figure, that is, a line as long as all its sides together, is called the *perimeter* of that figure. The above proportion may therefore be expressed in shorter terms; namely, the perimeters of two regular polygons of the same number of sides, are to each other in the proportion of the radii of the inscribed or circumscribed circles.

2. *The areas of two regular polygons of the same number of sides, are in the same ratio as the squares constructed upon the radii of the inscribed or circumscribed circles. Thus the area of the regular polygon ABCDEF, is as many times greater than the area of the regular polygon* abcdef, *as the area of the square upon the radius OA is greater than the area of the square upon the radius* oa. For the areas of the similar triangles ABO, *abo*, are to each other as the squares upon the corresponding sides (the radii OA, *oa*); therefore, any number of times (in our figure 6 times) the areas of these triangles, that is, the areas of the regular polygons ABCDEF, *abcdef*, themselves, are to each other in the same ratio. In the same manner I can prove that the area of the polygon ABCDEF is as many times greater than the area of the polygon *abcdef*, as the square upon

the circle *inscribed* in the regular polygon ABCDEF, is greater than the square upon the radius of the circle inscribed in the regular polygon *abcdef*.

* *

QUERY XXI.



From what you have learned of the properties of regular polygons, can you give a rule for finding the area of a regular polygon?

A. Yes. Multiply the sum of all the sides (the perimeter) by the radius of the inscribed circle; the product, divided by 2, will be the area of the regular polygon.

Q. Why?

A. Because every regular polygon, the polygon ABCDEF, for instance, can be divided into as many equal triangles, as there are sides in the polygon; and the area of each of these triangles is found by multiplying the basis, that is, one of the sides of the polygon by the height (which, in every one of these triangles, is equal to the radius, *om*, of the inscribed circle), and dividing the product by 2; therefore the area of the whole polygon ABCDEF may at once be found by multiplying the *sum* of all the sides by the radius of the inscribed circle, and dividing the product by 2.*

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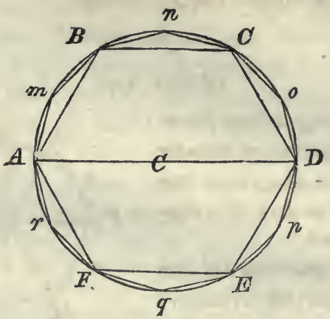
* Instead of multiplying the perimeter by the whole radius, and then dividing the product by 2, you may at once multiply the perimeter by *half* the radius, or the radius by half the perimeter.

QUERY XXII.



If you bisect each of the arcs AB, BC, CD, &c., subtended by the sides AB, BC, CD, &c., of a regular polygon inscribed in a circle; and then to the points of division, m, n, o, p, q, r, *draw the lines A*m, m*B*, *B*n, n*C*, *C*o, *&c.; what do you observe with regard to the regular polygon, A*m*B*n*C*o*D*p*E*q*F*r, *thus inscribed in the circle?*

*A. The regular polygon A*m*B*n*C*o*D*p, *&c., has twice as many sides as the regular polygon ABCDEF;* for the circumference of the circle is now divided into twice as many equal parts as before. Thus if the regular polygon ABCDEF has 6 sides, the regular polygon A*m*B*n*C*o*D*p*, &c., has 12 sides; and by bisecting again the arcs A*m*, *m*B, B*n*, &c., I can inscribe a regular polygon of 24 sides, and so on, by continuing to bisect the arcs, a regular polygon of 48, 96, 192, &c., sides.

Q. And what do you observe with regard to the arcs which are subtended by the sides of the polygons, ABCDEF and A*m*B*n*C*o*D*p*E*q*F*r*, inscribed in the circle?

A. The arcs, AB, BC, CD, &c., subtended by the sides of the regular polygon ABCDEF, first inscribed in

the circle, stand farther off the sides AB, BC, CD, &c., than the arcs A*m*, *m*B, B*n*, &c., from the sides A*m*, *m*B, B*n*, &c., of the regular polygon of twice the number of sides; consequently, if the arcs AB, BC, CD, &c., were drawn out into straight lines, they would differ more from the sides AB, BC, CD, &c., of the regular polygon ABCDEF, first inscribed in the circle, than the arcs A*m*, *m*B, B*n*, *n*C, &c., would, in this case, differ from the sides A*m*, *m*B, B*n*, &c., of the regular polygon A*m*B*n*C*o*, &c., of twice the number of sides.

Q. Now, if, continuing to bisect the arcs, you inscribe regular polygons of 24, 48, 96, 192, &c. sides, what further remark can you make with regard to the arcs subtended by the sides of *these* polygons?

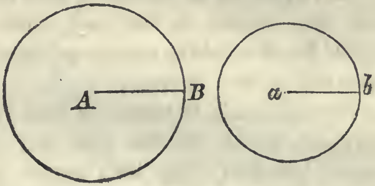
A. These arcs differ less in length from the sides which subtend them, in proportion as the polygon consists of a greater number of sides; because, by continuing to bisect the arcs, and thereby to increase the number of sides of the inscribed polygons, the arcs subtended by these sides grow nearer and nearer to the sides themselves, and finally the difference between them will become imperceptible.

Q. *And what conclusion can you now draw respecting the whole circumference of a circle?*

A. *That the circumference of a circle differs very little from the sum of all the sides of a regular inscribed polygon of a great number of sides; therefore, if the number of sides of the inscribed polygon is very great (several thousand for instance), the polygon will differ so little from the circle itself, that, without perceptible error, the one may be taken for the other.*

QUERY XXIII.

It has been shown in the last query, that a circle may be considered as a regular polygon of a very great number of sides; what inferences can you now draw with regard to the circumferences and areas of circles?



A. 1. *The circumferences of two circles are in proportion to the radii of these circles; that is, a straight line as long as the circumference of the first circle, is as many times greater than a straight line as long as the circumference of the second circle, as the radius AB of the first circle, is greater than the radius* ab *of the second circle.* For if, in each of the two circles, a regular polygon of a very great number of sides is inscribed, the sums of all the sides of the two polygons are to each other in proportion to the radii, AB, *ab*, of the circumscribed circles (page 122, 1st); and as the difference between the circumference of a circle and the sum of all the sides of an inscribed polygon of a great number of sides, is imperceptible (last Query), we may say that the circumferences themselves are in the same ratio.*

* The teacher may give an ocular demonstration of this principle, by taking two circles, cut out of pasteboard or wood; and measuring their circumferences by passing a string around them. The measure of the one will be as many times greater than the measure of the other, as the radius of the first circle is greater than the radius of the second circle.

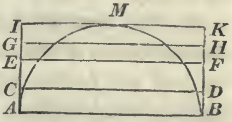
2. *The areas of two circles are in proportion to the squares constructed upon their radii; that is, the area of the greater circle is as many times greater than the area of the smaller circle, as the area of the square upon the radius of the greater circle, is greater than the area of the square constructed upon the radius of the smaller circle.* For if in each of these circles a regular polygon of a great number of sides is inscribed, the difference between the areas of the polygons and the areas of the circles themselves will be imperceptible; and because the areas of two regular polygons of the same number of sides are in the same ratio as the areas of the squares upon the radii of the circles in which they are inscribed (page 122, 2dly), the areas of the circles themselves are in the ratio of the squares upon their radii.

3. *The area of a circle is found by multiplying the circumference of the circle, given in rods, feet, inches, &c., by half the radius, given in units of the same kind.* Because a circle differs so little from a regular inscribed polygon of a great number of sides, that the area of the polygon may, without perceptible error, be taken for the area of the circle. Now the area of a regular polygon inscribed in a circle, is found by multiplying the sum of all the sides by the radius of the *inscribed* circle, and dividing the product by 2 (page 123); therefore the area of the circle itself is found by multiplying the circumference (instead of the sums of all the sides of the inscribed polygon) by the radius, and dividing the product by 2. For it has been shown in the last Query, that the sides of a regular inscribed polygon grow nearer and nearer the circumference of the circumscribed circle, in proportion as these sides increase in number; consequently, the circumference of a circle *inscribed* in a regular polygon of a great number of sides, will also grow nearer and

nearer the circumference of the *circumscribed* circle; until finally the two circumferences will differ so little from each other, that the radius of the one may, without perceptible error, be taken for the radius of the other.

Remark. Finding the *area of a circle* is sometimes called *squaring the circle.* The problem to construct a *rectilinear figure,* for instance a rectangle, whose area shall exactly equal the area of a given circle, is that which is meant by finding the *quadrature of the circle.* For the area of any geometrical figure, terminated by straight lines only, can easily be found by the rule given in Query 5, Sect. III; or, in other words, we can always construct a square, which shall measure exactly as many square rods, feet, inches, &c., as a rectilinear figure of any number of sides.

Now it is easy to show, that there is nothing absurd in the idea of constructing a rectilinear figure, for instance a rectangle, whose area shall be equal to the area of a given circle. For let us take a semicircle ABM, and let us for a moment imagine the diameter AB to move parallel to itself between the two perpendiculars AI, BK. It is evident that when the diameter AB is very near its original position, for instance in CD, the area of the rectangle ABCD is *smaller* than the area of the semicircle ABM; but the diameter continuing to move parallel to itself in the direction from A to I, there will be a point in the line AI, where the area of the rectangle ABIK is *greater* than the area of the semicircle ABM. Now as there is a point in the line AI, below the point I, in which the area of the rectangle ABCD is smaller than the area of the semicircle ABM, and as the diameter, by continuing to move in the same direction, makes in different points C, E, G, &c., of that same line, the rectangles ABCD, ABEF, ABGH, &c., whose areas become greater and greater, until finally they become greater than the area of the semicircle itself; there must evidently be a point in the line AI, in which a line drawn parallel to the diameter AB, makes with it and the perpendiculars AI, BK, a rectangle, which, in area, is *equal* to the semicircle ABM; and as there is a rectangle which, in area, is equal to the *semicircle* ABM, by doubling it, we shall have a rectangle which, in area, is equal to the *whole* circle.



Neither is it difficult to find the area of a circle *mechanically.* For the area of a circle being found by multiplying the circumference by the length of the radius, and dividing the product by 2 (page 127, 3dly), we need only pass a string around the circumference of a circle, and then multiply the length of that string by the length of the radius; the product divided by 2 will be the area of the circle. Having thus found the comparative length of the radius and circumference of one circle, we might determine the circumference, and thereby the area of any other circle, when knowing its radius. For the circumferences of two circles being in proportion to the radii of the two circles, we should have three terms of a geometrical proportion given; *viz.* the radii of the two circles, and the circumference of the one; from which we might easily find the fourth term (Theory of Proportions, page 64, 8thly), which would be the circumference of the other circle.

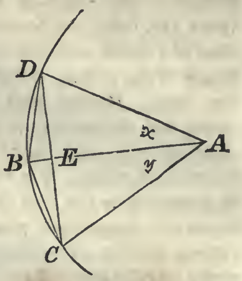
But the expressions of the circumference and area of a circle, thus obtained by measurement, are never so correct as is required for very nice and accurate mathematical calculations; we must therefore resort to other means, such as geometry itself furnishes, to *calculate* the ratio of the radius or diameter to the circumference; and herein consists the difficulty of the quadrature of the circle. For if the ratio of the radius to the circumference is once determined, we can easily find the circumference of any circle, when its radius is given; and knowing the circumference and the radius, we can find the area of the circle.

To calculate the ratio of the *diameter* to the circumference, mathematicians have compared the circumference of a circle to the sum of all the sides of a regular inscribed polygon of a great number of sides; for it has been shown (page 125), that the circumference of a circle differs very little from the sum of all the sides of such a polygon.

For this purpose they took a regular inscribed *hexagon*, each of the sides of which is equal to the radius of the circumscribed circle. (Query 17, Sect. IV.) For the sake of convenience they supposed the diameter of the circle equal to unity; the radius and therefore the side of a regular inscribed hexagon is then $\frac{1}{2}$, and the sum of all the sides (6 times $\frac{1}{2}$) equal to 3. This is the first approximation to the circumference of a circle.

From the side of a regular inscribed hexagon, it is easy to find that of a regular inscribed polygon of 12 sides. Supposing, for

instance, the chord CD to be the side of a regular inscribed hexagon, by bisecting the arc CD in B, the chords BC, BD, will be two sides of a regular inscribed polygon of 12 sides, the length of which can easily be calculated when the chord CD and the radius AC are once known. For the radius AB which bisects the arc CD, makes the angles x and y, which are measured by the arcs DB, BC, equal to each other; and therefore AE is perpendicular to the chord DC, and bisects it in E. (Page 104, 4thly.)



Now the radius, AC and EC (half of CD), being known, the hypothenuse and one of the sides of the right-angled triangle AEC are given, whence it is easy to find the other side AE, by the rule given in the remark, page 95. Thus if the radius is supposed to be $\frac{1}{2}$, the side CD of the inscribed hexagon is also equal to $\frac{1}{2}$; and EC (half of CD) is $\frac{1}{4}$. Taking the square of $\frac{1}{4}$ from that of $\frac{1}{2}$, and extracting the square root of the remainder, we obtain the length of the side AE, which, subtracted from the radius AB, leaves the length of BE. Now we can find the side BC in the right-angled triangle BCE, by extracting the square root of the *sum* of the squares of BE and EC (see the remark, page 95); and one of the sides of the regular inscribed polygon of 12 sides being once determined, we need only multiply it by 12, in order to obtain the sum of *all* its sides, which is the *second* approximation to the circumference of the circle. In precisely the same manner can the side, and consequently also the *sum* of all the sides of a regular inscribed polygon of 24 sides be obtained, when that of a regular inscribed polygon of 12 sides is once known; which is the third approximation to the circumference. Thus we might go on finding the sum of all the sides of a regular inscribed polygon of 48, 96, 192, &c., sides, until the inscribed polygon should consist of several thousand sides: the sum of all the sides would then differ so little from the circumference of the circle, that, without perceptible error, we might take the one for the other.

In this manner the approximation to the circumference of the circle has been carried further than is ever required in the minutest and most accurate mathematical calculations.

The beginning of this extremely tedious calculation gives the following results:

Parts of the circumference.	Sides of the inscribed polygon.	Sum of all the sides of the inscribed polygon.
6	0,5	3
12	0,258819	3,105828
24	0,130526	3,132628
48	0,065403	3,139348
96	0,032719	3,141033
192	0,016361	3,141446

It is not necessary to carry this calculation any further, since analysis furnishes us with means to obtain the same results in a much easier manner.

In nearly the same manner has LOUDOLPH VAN CEULEN found the ratio of the diameter to the circumference of a circle to 32 decimals. (See his 'Arithmetische en Geom. Fundamenten, page 163. Leiden. 1616;' also his work 'De Circulo et Adscriptis, c. 10. Leiden. 1619.'*)

ARCHIMEDES found the ratio of the diameter to the circumference as near 7 to 22.

FRANCISCUS VIETA found it as 1 to 3,1415926535.

ADRIANUS ROMANUS added the following decimals

89793.

LOUDOLPH VAN CEULEN added further

23846264338327950288.

SHARP added again

41971693993751058209749445923078,

To which MACHIN further added

16406286208998628034825342117067\9,

And lastly LAGNY increased them by

8214808651328230664709384\46.

In a manuscript in the library at Oxford, this number is still further extended by 29 decimals, namely,

46095505182231725359408128480\2.

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* The first work which LOUDOLPH VAN CEULEN published on this subject, bears the title 'Van den Circkel, daer in Gheleert wird te vinden de naeste proportie des Cirkels-Diameter tegen synen Omloop. Leiden. d. 20 Sept. 1596.' The work is dedicated to Prince MORIZ OF ORANGE

So that the most accurate ratio of the diameter to the circumference is at present as

1 to 3,1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132823066470938446095505822317253594081284802.

The last ratio is so near the truth, that in a circle, whose diameter is one hundred million times greater than that of the sun, the error would not amount to the one hundred millionth part of the breadth of a hair.

In general, when the calculations need not be very minute and accurate, 7 decimals will suffice. Thus we may consider the ratio of the diameter to the circumference to be

as 1 to 3,1415926; that is,

if the diameter of a circle is 1, its circumference is 3,1415926;* consequently if the diameter is 2, or the radius 1, the circumference will be twice 3,1415926, equal to 6,2831852. Dividing this number by 360, we obtain the length of a degree; dividing the length of a degree by 60, we obtain the length of a minute; and that again divided by 60, gives the length of a second, and so on. In this manner we obtain the length of

1 degree	equal	to	0,0174533†
1 minute	"	"	0,0002909
1 second	"	"	0,0000048
1 third	"	"	0,0000001

Having once determined the circumference of the circle whose radius is 1, we can easily find the circumference of any circle, when its radius is given; for we need only multiply the number 6,2831852 (that is, the circumference of a circle whose radius is 1), by the radius of the circle whose circumference is to be found; the product will be the circumference sought. Thus if it is required to find the circumference of a circle whose radius is 6 inches, we need only multiply the number 6,2831852 by 6; the product 37,6991112 is the circumference of that circle.

If it be required to find the length of an arc of a given number

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* The number 3,1415926 is sometimes represented by the Greek letter π. Thus the circumference of a circle, whose radius is 1, may be represented by 2π.

† The last figure in these expressions has been corrected.

of degrees, minutes, seconds, &c., in a given circle, we need only multiply

the degrees by 0,0174533
the minutes by 0,0002909
the seconds by 0,0000048, &c ;

the different products added together give the length of an arc of the same number of degrees, minutes, seconds, &c., in the circle whose radius is 1 ; and multiplying this product by the radius of the given circle, we shall have the length of the arc sought. If, for instance, it be required to find the length of an arc of 6 degrees and 2 minutes, in a circle whose radius is 5 inches, we in the first place multiply 0,0174533 by 6, and

0,0002909 by 2 ; the products of these multiplications, $\left.\begin{array}{r}0{,}1047198\\ \text{and } \underline{0{,}0005818}\end{array}\right\}$ added together,

give 0,1053016, which is the length of an arc of 6 degrees and 2 minutes in the circle whose radius is 1, and this last product (0,1053016), multiplied by 5, gives 0,5265080, which is the length of an arc of the same number of degrees and minutes of a circle whose radius is 5 inches.

Now that we are able to find the circumference, or an arc of the circumference of any circle, when knowing its radius, nothing can be easier than to calculate the area of a circle, of a sector, a segment, &c.

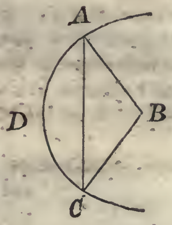
The area of a circle being found by multiplying the circumference by half the radius, or by multiplying half the circumference by the whole radius (page 127, 3dly), we need only take the number 3,1415926, which is half the circumference of the circle whose radius is 1, and multiply it by the radius of the given circle ; the product will be half the circumference of the given circle, which multiplied again by the radius, gives us the area of it. Thus if it is required to find the area of a circle whose radius is 5 inches, we multiply the number 3,1415926 twice in succession by 5, that is, we multiply it by the *square* of 5 ;* the product 78,5398150 is the area sought. Hence follows the general rule :

In order to find the area of a circle, multiply the number 3,1415926 *by the square of the radius.*

---

* Multiplying a number twice in succession by 5, is the same as multiplying that number by 25 ; which is the square of 5 ; because 5 times 5 are 25.

If the radius is given in rods, the answer will be square rods; if given in feet, the answer will be square feet, if in seconds, square seconds, and so on. The area of a semicircle is found by dividing the area of the whole circle by 2. In the same manner we find the area of a quadrant by dividing the area of the whole circle by 4, &c.



The area of a sector BCAD is found by multiplying the length of the arc CDA by half the radius, or we may first find what part of the circumference the arc CDA is; whether a third, a fourth, a fifth, &c., and then divide the area of the whole circle whose radius is BC, by 3, 4, 5, &c., according as the arc CDA is $\frac{1}{3}$, $\frac{1}{4}$, &c., of the whole circumference. If we are to find the area of the segment CDA, we must first find the area of the sector BCDA; then the area of the triangle ABC; which, subtracted from the area of the sector BCDA, will leave the area of the segment CDA.

RECAPITULATION OF THE TRUTHS CONTAINED IN THE FOURTH SECTION.

Q. Can you now repeat the different relations which exist between the different parts of a circle and the straight lines, which cut or touch the circumference?

A. 1. A straight line can touch the circumference only in one point.

2. When the distance between the centres of two circles is less than the sum of their radii, the two circles cut each other.

3. When the distance between the centres of two circles is equal to the sum of their radii, the two circles *touch* each other *exteriorly*.

4. When the distance between the centres of two

circles is equal to the difference between their radii, the two circles *touch* each other *interiorly*.

5. When two circles are concentric, that is, when they are both described from the same point as a centre, the circumferences of the two circles are parallel to each other.

6. A perpendicular, let fall from the centre of a circle, upon one of the chords in that circle, divides that chord into two equal parts.

7. A straight line, drawn from the centre of a circle to the middle of a chord, is perpendicular to that chord.

8. A perpendicular, drawn through the middle of a chord, passes, when sufficiently far extended, through the centre of the circle.

9. Two perpendiculars, each drawn through the middle of a chord in the same circle, intersect each other at the centre.

10. The two angles which two radii, drawn to the extremities of a chord, make with the perpendicular let fall from the centre of the circle to that chord, are equal to one another.

11. If two chords in the same circle, or in equal circles, are equal to one another, the arcs subtended by them are also equal; and the reverse is also true; that is, if the arcs are equal to one another, the chords which subtend them are also equal.

12. The greater arc stands on the greater chord, and the greater chord subtends the greater arc.

13. The angles at the centre of a circle are to each other in the same ratio, as the arcs of the circumference intercepted by their legs.

14. If two angles at the centre of a circle are equal to one another, the arcs of the circumference, intercepted by their legs, are also equal; and the reverse is also true;

that is, if the two arcs intercepted by the legs of the two angles at the centre of a circle, are equal to one another, these angles are also equal.

15. Angles are measured by arcs of circles, described with any radius between their legs. The circumference is, for this purpose, divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes; each minute, again, into 60 equal parts, called seconds, &c.

16. The magnitude of an angle does not depend on the *length* of the arc intercepted by its legs; but merely on the number of degrees, minutes, seconds, &c., it measures of the circumference.

17. The circumference of a circle is the measure of 4 right angles; the semi-circumference that of 2 right angles; and a quadrant that of 1 right angle.

18. A straight line drawn at the extremity of the diameter or radius, perpendicular to it, touches the circumference only in one point, and is therefore a tangent to the circle.

19. A radius or diameter drawn to the point of tangent, is perpendicular to the tangent.

20. A line drawn through the point of tangent, perpendicular to the tangent, passes, when sufficiently far extended, through the centre of the circle.

21. The angle, formed by a tangent and a chord, is half of the angle at the centre, which is measured by the arc subtended by that chord; therefore the angle, formed by the tangent and the chord, measures half as many degrees, minutes, seconds, &c., as the angle at the centre.

22. The angle which two chords make at the circumference of a circle, is half of the angle made by two radii at the centre, having its legs stand on the

extremities of the same arc; therefore every angle, made by two chords at the circumference of a circle, measures half as many degrees, minutes, seconds, &c., as the arc intercepted by its legs.

23. If several angles at the circumference have their legs stand on the extremities of the same arc these angles are all equal to one another.

24. Parallel chords intercept equal arcs of the circumference.

25. If, from a point without the circle, you draw a tangent to the circle, and, at the same time, a straight line cutting the circle, the tangent is a mean proportional between that whole line, and that part of it which is without the circle.

26. If a chord cuts another *within* the circle, the two parts, into which the one is divided, are in the inverse ratio of the two parts, into which the other is divided.

27. If, from a point without a circle, two straight lines are drawn, cutting the circle, these lines are to each other in the inverse ratio of their parts without the circle.

28. If the circumference of a circle is divided into 3, 4, 5, &c., equal parts, and then the points of division are joined by straight lines, the rectilinear figure, thus inscribed in the circle, is a regular polygon of as many sides, as there are parts into which the circumference is divided.

29. If, from the centre of a regular polygon, inscribed in a circle, radii are drawn to all the vertices at the circumference, the angles which these radii make with each other at the centre, are all equal to one another.

30. The side of a regular hexagon inscribed in a circle, is equal to the radius of the circle.

31. If, from the centre of a circle, radii are drawn, bisecting the sides of a regular inscribed polygon, and

then, at the extremities of these radii, tangents are drawn to the circle, these tangents form with each other a regular *circumscribed* polygon of the same number of sides as the regular inscribed polygon.

32. Around every regular polygon a circle can be drawn in such a manner, that all the vertices of the polygon shall be at the circumference of the circle.

33. Two regular polygons of the same number of sides are similar figures.

34. The sums of all the sides of two regular polygons of the same number of sides, are to each other in the same ratio, as the radii of the inscribed or circumscribed circles.

35. The areas of two regular polygons of the same number of sides, are to each other as the areas of the squares constructed upon the radii of the inscribed or circumscribed circles.

36. The area of a regular polygon is found by multiplying the sum of all its sides by the radius of the inscribed circle, and dividing the product by 2; or we may at once multiply half the sum of all the sides by the radius of the inscribed circle, or half that radius by the sum of all the sides.

37. If the arcs subtended by the sides of a regular polygon, inscribed in a circle, are bisected, and chords drawn from the extremities of these arcs to the points of division, the new figure thus inscribed in the circle, is a regular polygon of twice the number of sides as the one first inscribed.

38. The circumference of a circle differs so little from the sum of all the sides of a regular inscribed polygon of a great number of sides, that, without perceptible error, the one may be taken for the other.

39. The circumferences of two circles are in proportion

to the radii of these circles; that is, a straight line, as long as the circumference of the first circle, is as many times greater than a straight line as long as the circumference of the second circle, as the radius of the one is greater than the radius of the other.

40. The areas of two circles are in proportion to the squares constructed upon their radii; that is, the area of the greater circle is as many times greater than the area of the smaller circle, as the area of the square upon the radius of the one is greater than the area of the square upon the radius of the other.

41. The area of a circle is found by multiplying the circumference, given in rods, feet, inches, &c., by half the radius, given in units of the same kind.

42. The circumference of a circle, whose radius is 1, is equal to the number 6,2831852; and the circumference of any other circle is found by multiplying the number 6,2831852 by the length of the radius.

43. The length of 1 degree in a circle, whose radius is 1, is equal to the number 0,0174533

The length of 1 minute	0,0002909
" " " 1 second	0,0000048
" " " 1 third	0,0000001

44. The length of an arc, given in degrees, minutes, seconds, &c., is found by multiplying the degrees by 0,0174533, the minutes by 0,0002909, the seconds by 0,0000048, &c., then adding these products together, and multiplying their sum by the radius of the circle.

45. The area of a circle, whose radius is 1, is equal to 3,1415926 square units; and the area of any other circle is found by multiplying the number 3,1415926 by the square of the radius.

46. The area of a semicircle is found by dividing the area of the whole circle by 2.

47. The area of a quadrant is found by dividing the area of the whole circle by 4.

48. The area of a sector is found by multiplying the length of the arc by half the radius.

49. In order to find the area of a segment, we first draw two radii to the extremities of the arc of that segment; then calculate the area of the sector, formed by the two radii and that arc, and subtract from it the area of the triangle formed by the two radii and the chord of the segment; the remainder is the area of the segment.

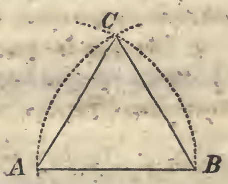
SECTION V.

APPLICATION OF THE FOREGOING PRINCIPLES TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

PART I.

Problems relative to the drawing and division of lines and angles.

Problem I. *To construct an equilateral triangle upon a given straight line, AB.*



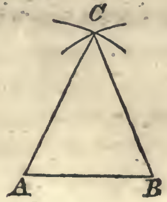
Solution. Let AB be the given straight line.

1. From the point A, as a centre, with the radius AB, describe an arc of a circle, and from the point B, with the same radius, AB, another arc cutting the first.

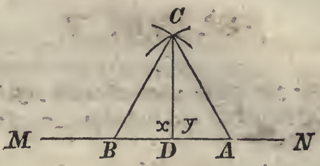
2. From the point of intersection, C, draw the lines AC, BC; the triangle ABC will be equilateral.

Demonstration. The three sides, AB, AC, BC, of the triangle ABC, are all equal to each other; because they are radii of equal circles.

Remark. In a similar manner can an isosceles triangle be constructed upon a given basis.



PROBLEM II. *From a given point in a straight line, to erect a perpendicular upon that line.*



I. SOLUTION. Let MN be the given straight line, and D the point in which the perpendicular is to be erected.

1. Take any distance, BD, on one side of the point D, and make DA equal to it.

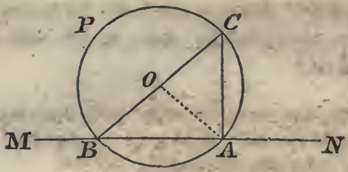
2. From the point B, with any radius greater than BD, describe an arc of a circle, and from the point A, with the same radius, another arc, cutting the first.

3. Through the point of intersection, C, and the point D, draw a straight line, CD, which will be perpendicular to the line MN.

DEMON. The three sides of the triangle BCD, are equal to the three sides of the triangle ACD, each to each, viz.

the side BC equal to AC
" " BD " " DA
" " CD " " CD;

therefore the three angles in the triangle BCD are also equal to the three angles of the triangle ADC, each to each (page 40); and the angle x opposite to the side BC in the triangle BCD, is equal to the angle y opposite to the equal side AC in the triangle ACD; and as the two adjacent angles, which the line CD makes with the line MN, are equal to one another, the line CD is perpendicular to MN. (Definitions of perpendicular lines, page 12.)



II. Solution. Let MN be the given straight line, and A the point in which the perpendicular is to be drawn to it.

1. From a point, O, as a centre, with a radius, OA, greater than the distance O from the straight line MN, describe the circumference of a circle.

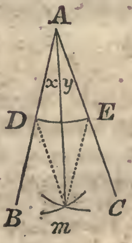
2. Through the point B and the centre O, of the circle, draw the diameter BC.

3. Through C and A draw a straight line, which will be perpendicular to the line MN.

Demon. The angle BAC, at the circumference, measures half as many degrees as the arc BPC intercepted by its legs (page 111, 1st). But the arc BPC is a semi-circumference; therefore the angle BAC, measures a quadrant; consequently the angle BAC is a right angle (page 107, Remark 3), and the line AC is perpendicular to MN.

Problem III. *To bisect a given angle.*

Solution. Let BAC be the given angle.



1. From the vertex, A, of the angle BAC, with a radius, AE, taken at pleasure, describe an arc of a circle; and from the two points D and E, where this arc cuts the legs of the given angle, with the same radius describe two other arcs, cutting each other in the point *m*.

2. Through the point m, and the vertex of the given angle, draw a straight line, Am, which will bisect the given angle BAC.

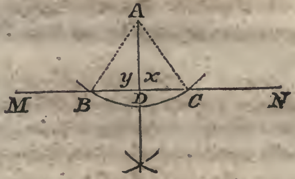
DEMON. The two triangles AmD, AmE, have the three sides in the one equal to the three sides in the other, viz.

the side AD = to the side AE
" " mD = " " " mE
" " Am = " " " Am;

consequently these two triangles are equal to each other; and the angle x, opposite to the side mD, in the triangle AmD, is equal to the angle y, opposite to the equal side mE, in the triangle AmE; therefore the angle BAC is bisected.

PROBLEM IV. *From a given point* without *a straight line, to let fall a perpendicular upon that line.*

SOLUTION. Let A be the given point, from which a perpendicular is to be drawn to the line MN.



1. With any radius sufficiently great describe an arc of a circle.

2. From the two points B and C, where this arc cuts the line MN, draw the straight lines BA, CA.

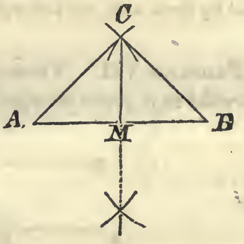
3. Bisect the angle BAC (see the last Problem), the line AD is perpendicular to the line MN.

DEMON. The two triangles ABD, ACD, have two sides, AB, AD, in the one, equal to two sides, AC, AD, in the other, each to each (AC, AB, being radii of the same circle, and the side AD being common to both); and have the angles included by these sides also equal (because the angle BAC is bisected); therefore these two triangles are equal to one another (Query 1, Sect. II.); and the angle y, opposite to the side AB, in the triangle ABD, is equal to the angle x, opposite to the equal side AC, in the triangle ACD. Now, as the two adjacent angles x and y, which the straight

line AD makes with the straight line MN, are equal to each other, the line AD must be perpendicular to MN. (Def. of perpendicular lines.)

PROBLEM V. *To bisect a given straight line.*

SOLUTION. Let AB be the given straight line.



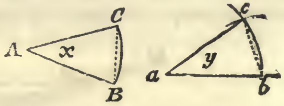
1. From A, with a radius greater than half of AB, describe an arc of a circle; and from B, with the same radius, another, cutting the first in the point C.

2. From the point C draw the perpendicular CM, and the line AB is bisected in M.

DEMON. The two right-angled triangles AMC, BMC, are equal, because the hypothenuse AC and the side CM in the one, are equal to the hypothenuse and the side CM in the other (page 47); and therefore the third side AM in the one, is also equal to the third side BM in the other; consequently the line AB is bisected in the point M.

PROBLEM VI. *To transfer a given angle.*

SOLUTION. Let x be the given angle, and A the point to which it is to be transferred.



1. From the vertex of the given angle, as a centre, with a radius taken at pleasure, describe an arc of a circle between the legs AB, AC.

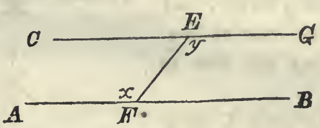
2. From the point *a*, as a centre, with the same radius, describe another arc, *cb*.

3. Upon the last arc take a distance, *bc*, equal to the chord BC.

4. Through *a* and *c* draw a straight line; the angle *y* is equal to the angle *x*.

Demon. The arcs BC, *bc*, are, by construction, equal to one another; therefore the angles *x* and *y*, at the centre, being measured by these arcs, are also equal to one another (page 106, 1st).

Problem VII. *Through a given point draw a line parallel to a given straight line.*

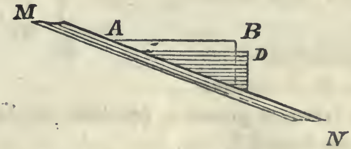


Solution. Let E be the point, through which a line is to be drawn parallel to the straight line AB.

1. Take any point, F, in the straight line AB, and join EF.

2. In E make the angle *y* equal to the angle *x*; the line EG, extended, is parallel to the line AB.

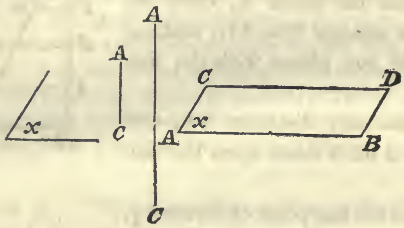
Demon. The two straight lines CG, AB, are cut by a third line EF, so as to make the alternate angles *x* and *y* equal; therefore these two lines are parallel to each other (page 29, 2dly).



Mechanical Solution. Take a ruler, MN, and put it in such a position that a right-angled triangle, passing along its edge, as you see in the figure, will make with it, in different points, A, C, &c., the lines AB, CD, &c.

These lines are parallel to each other, because they are cut by the edge of the ruler at equal angles.*

Problem VIII. *Two adjacent sides and the angle included by them being given, to construct a parallelogram.*



Solution. Let AB and AC be the two sides of the parallelogram, and x the angle included by them.

1. Make an angle equal to x.

2. Make the leg AB of that angle equal to AB, and the leg AC equal to AC.

3. Through the point C draw CD parallel to AB, and through B, the line BD parallel to AC; the quadrilateral ABCD is the required parallelogram.

Demon. The opposite sides of the quadrilateral ABCD, are parallel to each other; therefore the figure is a parallelogram. (See Def. page 13.)

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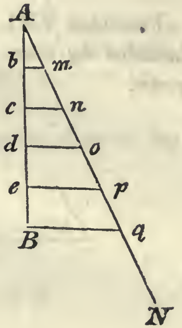
* This is a better way of drawing parallel lines than the common method by a parallel ruler, which is seldom very accurate, on account of the instrument being frequently out of order, and the great steadiness of hand required in the use of it.

Problem IX. *To divide a given line into any number of equal parts.*



I. **Solution.** Let AB be the given line, and let it be required to divide it into five equal parts.

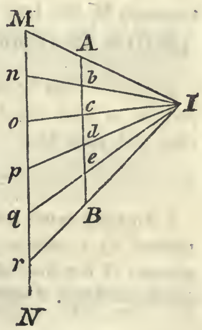
1. From the point A, draw an indefinite straight line AN, making any angle you please with the line AB.

2. Take any distance A*m*, and measure it off 5 times upon the line AN.

3. Join the last point of division *q*, and the extremity B of the line AB.

4. Through *m*, *n*, *o*, *p*, *q*, draw the straight limes *bm*, *cn*, *do*, *ep*, parallel to B*q*; the line AB is divided into five equal parts.

The demonstration follows immediately from Query 14, Sect. II.



II. **Solution.** Let AB be the given straight line, which is to be divided into 5 equal parts.

1. Draw a straight line MN, greater than AB, parallel to AB.

2. Take any distance M*n*, and measure it off 5 times upon the line MN.

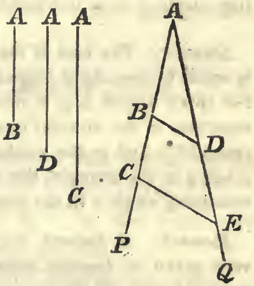
3. Join the extremities of both the lines M*r* and AB, by the straight lines MA, *r*B, which will cut each other, when sufficiently extended, in a point I.

4. Join I*n*, I*o*, I*p*, I*q*, the line AB is divided into 5 equal parts, viz. A*b*, *bc*, *cd*, *de*, *e*B.

Demon. The triangles A*b*I, *bc*I, *cd*I, *de*I, *e*BI, are similar to the triangles M*n*I, *no*I, *op*I, *pq*I, *qr*I, each to each; because the line AB is drawn *parallel* to M*r* (Query 16, Sect. II.); and as the bases M*n*, *no*, *op*, *pq*, *qr*, of the latter triangles are all equal to one another, the bases A*b*, *bc*, *cd*, *de*, *e*B of the former triangles must also be equal to one another.

Remark. If it were required to divide a line into two parts which shall be in a given ratio, for instance, as 2 to 3, you need only, as before, take 5 equal distances upon the line MN, and then join the point I to the second and last point of division; the line AB will, in the point *c*, be divided in the ratio of 2 to 3. In a similar manner can any given straight line be divided into 3, 4, 5, &c. parts, which shall be to each other in a given ratio.

Problem X. *Three lines being given, to find a fourth one, which shall be in a geometrical proportion with them.*



Solution. Let AB, AC, AD, be the given straight lines, which are three terms of a geometrical proportion, to which the fourth term is wanting. (See Theory of Proportions, Principle 8th, page 64.)

1. Draw two indefinite straight lines AP, AQ, making with one another any angle you please.

2. Upon one of these lines measure off the two distances AB, AC, and on the other the distance AD.

3. Join BD, and through C draw CE parallel to BD; the line AE is the fourth term in the geometrical proportion

$$AB : AC = AD : AE.$$

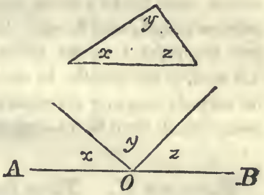
Demon. The triangle ABD is similar to the triangle ACE, from which it may be considered as cut off by the line BD being

drawn parallel to CE (Query 16, Sect. II.); and as in similar triangles the corresponding sides are in a geometrical proportion (page 70, 4thly), we have

$$AB : AC = AD : AE.$$

Problem XI. *Two angles of a triangle being given, to find the third one.*

Solution. Let x and y be the two given angles of the triangle, and let it be required to find the third angle z.



In any point O of an indefinite straight line AB, make two angles x and y, equal to the two given angles of the triangle; the remaining angle z is equal to the angle z in the triangle.

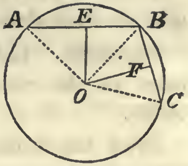
Demon. The sum of the three angles x, z, y, in the triangle, is equal to two right angles (Query 13, Sect. I.), and the sum of the three angles x, y, z, made in the same point O, and on the same side of the straight line AB, is also equal to two right angles (page 23); and as the angles x and y are made equal to the angle x and y in the triangle, the remaining angle z is also equal to the remaining angle z in the triangle.

Remark. If, instead of the angles themselves, their measure were given in degrees, minutes, seconds, &c., you need only subtract the sum of the two angles from 180 degrees, which is the measure of two right angles; the remainder is the angle sought.

Problem XII. *Through three given points, which are not in the same straight line, to describe the circumference of a circle.*

Solution. Let A, B, C, be the three points, through which it is required to pass the circumference of a circle.

1. Join the three points A, B, C, by the straight lines AB, BC.



2. Bisect the lines AB, BC.

3. In the points of bisection E and F, erect the perpendiculars EO, FO, which will cut each other in a point O.

4. From the point O as a centre, with a radius equal to the distance AO, describe the circumference of a circle, and it will pass through the three points A, B, C.

Demon. The two points A and B are at an equal distance from the foot of the perpendicular EO; therefore AO and BO are equal to one another (page 45, 5thly); for the same reason is BO equal to OC; because the points B and C are at an equal distance from the foot of the perpendicular FO; and as the three lines AO, BO, CO are equal to one another, the three points A, B, C, must necessarily lie in the circumference of the circle described with the radius AO.

Problem XIII. *To find the centre of a circle, or of a given arc.*

Solution. Let the circle in the last figure be the given one.

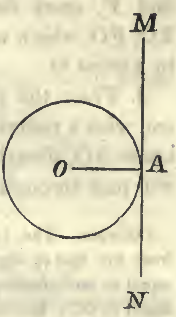
1. Take any three points A, B, C, in the circumference, and join them by the chords AB, BC.

2. Bisect each of these chords, and in the points of bisection erect the perpendiculars EO, FO; the point O, in which these perpendiculars meet each other, is the centre of the circle.

In precisely the same manner can the centre of an arc be found.

The demonstration is exactly the same as in the last problem.

Problem XIV. *In a given point in the circumference of a circle, to draw a tangent to that circle.*

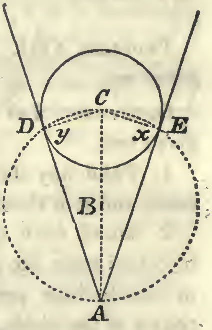


Solution. Let A be the given point in the circumference of the circle.

Draw the radius AO, and at the extremity A, perpendicular to it, the line MN; and it is a tangent to the given circle.

Demon. The line MN being drawn at the extremity AO of the radius, and perpendicular to it, touches the circumference in only one point (page 108)

Problem XV. *From a given point* without *a circle, to draw a tangent to the circle.*



Solution. Let A be the given point, from which a tangent is to be drawn to the circle.

1. Join the point A and the centre C, of the given circle.

2. From the middle of the line AC as a centre, with a radius equal to BC = AB, describe the circumference of a circle.

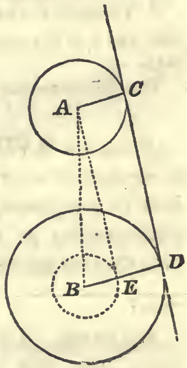
3. Through the points E and D, where this circumference cuts the circumference of the given circle, draw the lines AD, AE; and they are tangents to the given circle.

Demon. Join DC, EC. The angles x and y, being both angles at the circumference of the circle whose centre is B, measure each half as many degrees as the arc on which their legs stand. Both the angles, x and y, have their legs standing on the diameter AC, of the circle B; therefore each of these angles measures half

as many degrees as the semi-circumference (page 107, Rem. 3d); consequently, they are both right angles, and the lines AE and DA, being perpendicular to the radii CE, DC, are both tangents to the circle C.

Remark. From a point without a circle, you can always draw *two* tangents to the same circle.

PROBLEM XVI. *To draw a tangent common to* two *given circles.*



SOLUTION. Let A and B be the centres of the given circles, and let it be required to draw a tangent, which shall touch the two circles on the same side.

1. Join the centres of the two given circles by the straight line AB.

2. From B, as a centre, with a radius equal to the difference between the radii of the given circles, describe a third circle.

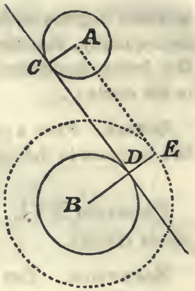
3. From A draw a tangent AE to that circle (see the last problem).

4. Draw the radius BE, and extend it to D.

5. Draw the radius AC parallel to BD.

6. Through C and D draw a straight line, and it will be a tangent common to the two given circles.

DEMON. The radius AC being equal and parallel to ED, it follows that ACED is a parallelogram; and because the tangent AE is *perpendicular* to the radius BE (page 108, 1st), CD is perpendicular to BD; consequently also to AC (because AC is parallel to BD); and the line CD, being perpendicular to both the radii AC, BD, is a tangent common to the two given circles.



If it be required to draw a tangent common to two given circles, which shall touch them on opposite sides, then

1. From B as a centre, with a radius equal to the *sum* of the radii of the given circles, describe a third circle.

2. From A draw a tangent AE to that circle.

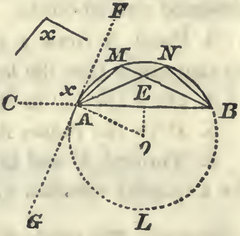
3. Join BE, cutting the given circle in D.

4. Draw AC parallel to BE.

5. Through C and D, draw a straight line, and it is the required tangent, touching the circles on opposite sides.

The demonstration is the same as the last.

Problem XVII. *Upon a given straight line to describe a segment of a circle, which shall contain a given angle; that is, a segment, such that the inscribed angles, having their vertices in the arc of the segment and their legs standing on its extremities, shall each be equal to a given angle.*



Solution. Let AB be the given line, and x the given angle.

1. Extend AB towards C.

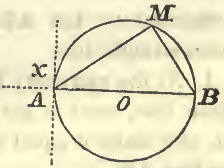
2. Transfer the angle x to the point A.

3. Bisect AB in E.

4. From the points A and E, draw the lines AO and EO, respectively, perpendicular to FG and CB.

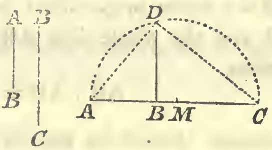
5. From the point O, the intersection of these perpendiculars, as a centre, with a radius equal to OA, describe a circle; AMNB is the required segment.

DEMON. The line FG being, by construction, perpendicular to the radius AO, is a tangent to the circle (page 108); and the angle GAB, formed by that tangent and the chord AB, is equal to either of the angles AMB, ANB, &c., that can be inscribed in the segment AMNB; because the angle GAB measures half as many degrees as the arc ALB (page 109), and each of the angles AMB, ANB, &c., at the circumference, having its legs standing on the extremities of the chord AB, measures also half as many degrees as the arc ALB (page 111); and as the angle GAB is equal to the angle x, GAB and x being opposite angles at the vertex (Query 5, Sect. I.), each of the angles AMB, ANB, &c., is also equal to the given angle x.



Remark. If the angle x is a right angle, the segment AMB is a semicircle, and the chord AB a diameter. To finish the construction, you need only from the middle of the line AB as a centre, with a radius equal to OA, describe a semicircle, and it is the required segment; for the angle AMB at the circumference measuring half as many degrees as the semicircumference AB, on which its legs stand, is a right angle.

PROBLEM XVIII. *To find a mean proportional* (see page 66) *to two given straight lines.*



SOLUTION. Let AB, BC, be the two given lines.

1. Upon an indefinite straight line, take the two distances AB, BC.

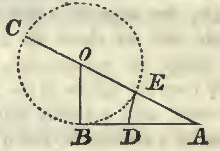
2. Bisect the whole distance AC, and from M, the middle of AC, with a radius equal to AM, describe a semi-circumference.

3. In B erect a perpendicular to the diameter AC, and extend it until it meets the semi-circumference in D; the line DB is a mean proportional between the lines AB and BC.

DEMON. The triangle ADC is right-angled in D; because the angle ADC is inscribed in a semicircle (see the remark to the last problem); and the perpendicular DB let fall from the vertex D, of the right angle upon the hypothenuse, is a mean proportional between the two parts AB, BC, into which it divides the hypothenuse, (page 75, 1st); therefore we have the proportion

$$AB : BD = BD : BC.$$

PROBLEM XIX. *To divide a given straight line into two such parts, that the greater of them shall be a mean proportional between the smaller part and the whole of the given line.*



SOLUTION. Let AB be the given straight line.

1. At the extremity B of the given line, erect a perpendicular, and make it equal to half of the line AB.

2. From O, as a centre, with a radius equal to OB, describe a circle.

3. Join the centre O of that circle, and the extremity A of the given line, by the straight line AO.

4. From AB cut off a distance AD equal to AE; then AD is a mean proportional between the remaining part BD, and the whole line AB; that is, you have the proportion

$$AB : AD = AD : BD.$$

DEMON. Extend the line AO until it meets the circumference in C. Then the radius OB, being perpendicular to the line AB, we have from the same point A, a tangent AB, and another line AC drawn cutting the circle; therefore we have the proportion

$$AC : AB = AB : AE;$$

for the tangent AB is a mean proportional between the whole line AC, and the part AE without the circle. (Query 13, Sect. IV.)

Now, in every geometrical proportion, you can add or subtract the second term once or any number of times from the first term,

and the fourth term the same number of times from the third term, without destroying the proportion (page 62, 6th). According to this principle you have

$$AC - AB : AB = AB - AE : AE;$$

that is, the line AC less the line AB, is to the line AB, as the line AB less EA, is to the line AE. But AC less AB is the same as the line AC less the diameter CE (because the *radius* of the circle is, by construction, equal to *half* the line AB); and AB less AE, is the same as AB less AD (because AD is made equal to AE); therefore you may write the above proportion also

$$AE : AB = BD : AE,\text{* or also}$$

$$AD : AB = BD : AD;$$

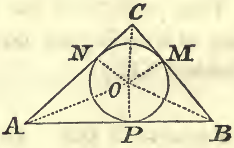
and because in every geometrical proportion the order of the terms may be changed in both ratios (Principle 1, of Geom. Prop.), you can change the last proportion into

$$AB : AD = AD : BD;$$

that is, the part AD of the line AB, is a mean proportional between the whole line AB and the remaining part BD.

Problem XX. *To inscribe a circle in a given triangle.*

Solution. Let the given triangle be ABC.



1. Bisect two of the angles of the given triangle; for instance the angles at C and B, by the lines CO, BO.

2. From the point O, where these lines cut each other, let fall a perpendicular upon any of the sides of the given triangle.

3. From O, as a centre, with the radius OP, equal to the length of that perpendicular, describe a circle, and it will be inscribed in the triangle ABC.

Demon. From O let fall the perpendiculars OM, ON, upon the two sides BC, AC, of the given triangle. The angle OCM is,

* AC less the diameter CE, being equal to AE; and BA less AD, equal to BD.

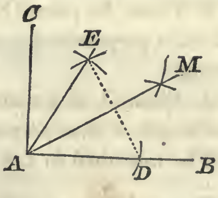
by construction, equal to the angle OCN (because the angle ACB is bisected by the line CO); and CMO, CNO, being right angles, the angles COM and CON are also equal to one another (because when two angles in one triangle are equal to two angles in another, the third angles in these triangles are also equal); therefore the two triangles CMO, CNO, have a side CO, and the two adjacent angles in the one, equal to the same side CO, and the two adjacent angles in the other; consequently these two triangles are equal to one another; and the side OM, opposite to the angle OCM in the one, is equal to the side ON, opposite to the equal angle OCN in the other. In the same manner it may be proved that the perpendicular OM is also equal to OP; and as the three perpendiculars OM, ON, OP, are equal to one another, the circumference of a circle described from the point O as a centre, with a radius equal to OP, passes through the three points M, N, P; and the sides AB, BC, AC of the given triangle, being perpendicular to the radii OP, OM, ON, are tangents to the inscribed circle (page 108).

PROBLEM XXI. *To circumscribe a circle about a triangle.*

This problem is the same as to make the circumference of a circle pass through three given points. (See Problem XII.)

PROBLEM XXII. *To trisect a right angle.*

SOLUTION. Let BAC be the right angle which is to be divided into three equal parts.



1. Upon AB take any distance AD, and construct upon it the equilateral triangle ADE. (Problem I.)

2. Bisect the angle DAE by the line AM (Problem III.); and the right angle BAC is divided into the three equal angles CAE, EAM, MAB.

Demon. The angle BAE being one of the angles of an equilateral triangle, is one third of two right angles (page 33), and therefore *two* thirds of *one* right angle; consequently CAE is *one* third of the right angle BAC; and since the angle BAE is bisected by the line AM, the angles EAM, MAB, are each of them also equal to one third of a right angle; and are therefore equal to the angle CAE and to each other.

PART II.

Problems relative to the transformations of geometrical figures.

Problem XXIII. *To transform a given quadrilateral figure into a triangle of equal area, whose vertex shall be in a given angle of the figure, and whose base in one of the sides of the figure.*

Fig. I.

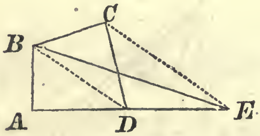
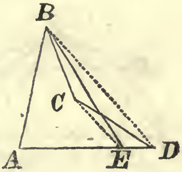


Fig. II.



Solution. Let ABCD (Fig. I. and II.), be the given quadrilateral; the figure I. has all its angles outwards, and the figure II. has one angle, BCD, inwards; let the vertex of the triangle, which shall be equal to it, fall in B.

1. Draw the diagonal BD (Fig. I. and II.), and from C, parallel to it, the line CE.

2. From E, where the line CE cuts AD (Fig. II.), or its further extension (Fig. I.), draw the line EB; the triangle ABE is equal to the quadrilateral ABCD.

Demon. The area of the triangle BCD (Fig. I. and II.) is equal to the area of the triangle BDE; because these two triangles are upon the same basis, BD, and between the same parallels, BD, CE (page 90, 3dly); consequently (Fig. I.), the *sum* of the areas of the two triangles ABD and BDC, is equal to the sum of the areas of the two triangles ABD, BDE; that is, the area of the quadrilateral ABCD is equal to the sum of the areas of the two triangles ABD, BDE, which is the area of the triangle ABE.

And in figure II. the *difference* between the areas of the two triangles ABD, BCD, that is, the quadrilateral ABCD, is equal to the difference between the triangles ABD, EBD, which is the triangle ABE.

Problem XXIV. *To transform a given pentagon into a triangle, whose vertex shall be in a given angle of the pentagon, and whose base upon one of its sides.*

Solution. Let ABCDE (Fig. I. and II.), be the given pentagon; let the vertex of the triangle, which is to be equal to it, be in C.

Fig. I.

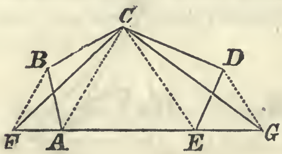
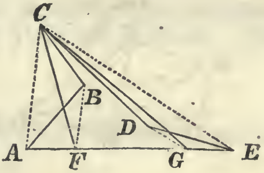


Fig. II.



1. From C draw the diagonals CA, CE.

2. From B draw BF parallel to CA, and from D draw DG parallel to CE.

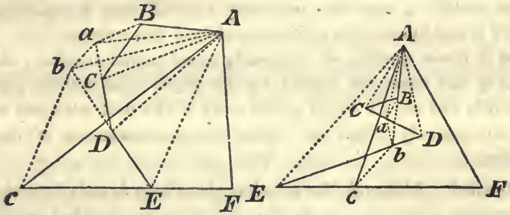
3. From F and G, where these parallels cut AE or its further extension, draw the lines CF, CG; CFG is the triangle required.

Demon. In both figures, we have the area of the triangle CBA equal to the area of the triangle CFA; because these two triangles are upon the same basis, CA, and between the same par-

allels, AC, FB; and for the same reason is the area of the triangle CDE equal to the area of the triangle CGE; therefore in figure I. the *sum* of the areas of the three triangles CAE, CBA, CDE, is equal to the sum of the areas of the triangles CAE, CFA, CGE; that is, the area of the pentagon ABCDE is equal to the area of the triangle CFG; and in figure II. the *difference* between the area of the triangle CAE and the areas of the two triangles CBA, CDE, is equal to the difference between the area of the same triangle CAE, and the areas of the two triangles CFA, CGE; that is, the area of the pentagon ABCDE is equal to the area of the triangle CFG.

Problem XXV. *To convert any given figure into a triangle, whose vertex shall be in a given angle of the figure, and whose basis shall fall upon one of its sides.*

Fig. I. Fig. II.



Let ABCDEF (Fig. I. and II.) be the given figure (in this case a hexagon), and A the angle in which the vertex of the required triangle shall be situated. For the sake of perspicuity, I shall enumerate the angles and sides of the figure from A, and call the first angle A, the second B, the third C, and so on; further, AB the first side, BC the second, DE the third, and so on. We shall then have the following general solution.

1. From A to all the angles of the figure, draw the diagonals AC, AD, AE, which, according to the order in which they stand here, call the first, second, and third diagonal.

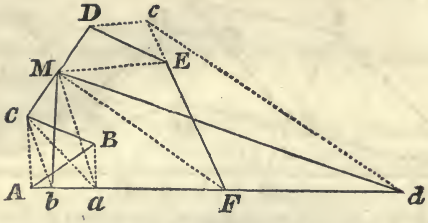
2. Draw from the second angle, B, a line, Ba, parallel to the first diagonal, AC; from the point where the parallel meets the third side, CD (Fig. II.), or its further extension (Fig. I.), draw a line, ab, parallel to the second diagonal, AD; and from the point b, where this meets the fourth side DE (Fig. II.) or its further extension (Fig. I.), draw another line, bc, parallel to the third diagonal.

3. When, in this way, you have drawn a parallel to every diagonal, then, from the last point of section of the parallels and sides (in this case c), draw the line cA; AcF is the required triangle, whose vertex is in A, and whose basis is in the side EF.

The demonstration is similar to the one given in the two last problems. First, each of the hexagons is converted into the pentagon AaDEF; then the pentagon AaDEF into a quadrilateral, AbEF; and finally this quadrilateral into the triangle AcF. The areas of these figures are evidently equal to one another; for the areas of the triangles, which, by the above construction, are successively cut off, are equal to the areas of the new triangles which are successively added on. (See the demonstration of the last problem.)

Remark. Although the solution given here is only intended for a hexagon, yet it may easily be applied to every other rectilinear figure. All depends upon the substitution of one triangle for another, by means of parallel lines. It is not absolutely necessary actually to draw the parallels; it is only requisite to denote the points in which they cut the sides, or their further extension; because all depends upon the determination of these points.

PROBLEM XXVI. *To transform any given figure into a triangle whose vertex shall be in a certain point, in one of the sides of the figure, or within it, and whose base shall fall upon a given side of the figure.*

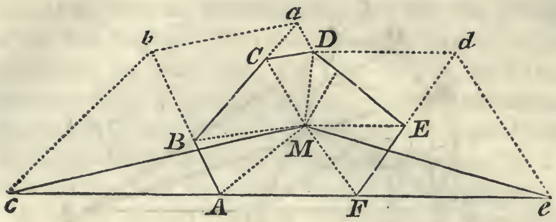


SOLUTION. 1*st Case.* Let ABCDEF be a hexagon, which is to be transformed into a triangle; let the vertex of the triangle be in the point M in the side CD, and the base in AF.

1. In the first place, get rid of the angle ABC, by drawing B*a* parallel to CA, and joining C*a;* the triangle CB*a*, substituted for its equal the triangle AB*a* (for these two triangles are upon the same basis, *a*B, and between the same parallels, CA, B*a*), transforms the hexagon ABCDEF into the pentagon *a*CDEF.

2. Draw the lines M*a*, MF, and the pentagon *a*CDEF is divided into three figures, *viz.* the triangle M*a*F, the quadrilateral MDEF on the right, and the triangle MC*a* on the left.

3. Transform the quadrilateral MDEF and the triangle MC*a* into the triangles M*d*F, M*ba*, so that the basis may be in AF (see the last problem); the triangle M*bd* is equal to the given hexagon.



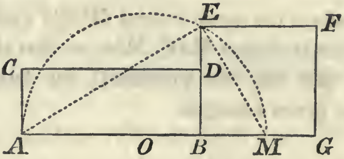
2d Case. Let ABCDEF be the given figure; let the vertex of the required triangle be situated in the point M within the figure, and let the base fall upon AF.

1. From M to any angle of the figure, say D, draw the line MD, and draw the lines MA, MF, by which means the figure ABCDEF is divided into the triangle MAF, and the figures MDCBA, MDEF.

2. Then transform MDCBA and MDEF into the triangles M*c*A, M*e*F, whose bases are in the continuation of AF; the triangle *c*M*e* is equal to the figure ABCDEF.

The demonstration follows from those of the last three problems.

PROBLEM XXVII. *To transform a given rectangle into a square of equal area.*



SOLUTION. Let ABCD be the given rectangle.

1. Extend the greater side, AB, of the rectangle, making BM equal to BD.

2. Bisect AM in O, and, from the point O as a centre, with a radius AO, equal to OM, describe a semicircle.

3. Extend the side BD of the rectangle, until it meets the circle in E.

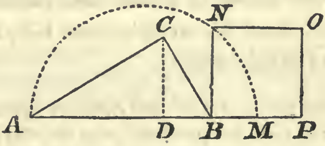
4. Upon BE construct the square BEFG, which is the square sought.

Demon. The perpendicular BE is a mean proportional between AB and BM (see Problem XVIII.); therefore we have the proportion

$$AB : BE = BE : BM;$$

and as, in every geometrical proportion, the product of the means equals that of the extremes (Theory of Prop., Principle 10, page 65), we have the product of the side BE multiplied by itself, equal to the product of the side AB of the parallelogram, multiplied by the adjacent side BD (or BM). But the first of these products is the area of the square BEFG, and the other is the area of the rectangle ABCD; therefore these two figures are, in area, equal to one another.

Problem XXVIII. *To transform a given triangle into a square of equal area.*



Solution. Let ABC be the given triangle, AB its base, and CD its height.

1. Extend AB by half the height CD.

2. Upon AM as a diameter, describe a semicircle.

3. From B draw the perpendicular BN, which is the side of the square sought.

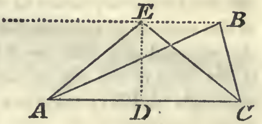
Demon. From the demonstration in the last problem, it follows, that the square upon BN is equal to the rectangle, whose base is AB, and whose height is BM (half the height of the triangle ABC). But the triangle ABC is equal to a rectangle upon the same base AB, and of half the height CD (page 89, 1st); therefore

the area of the square BNOP is equal to the area of the triangle ABC.

Remark. It appears from this problem, that every rectilinear figure can be converted into a square of equal area. It is only necessary to convert the figure into a triangle (according to the rules given in the problems 23, 24, 25), and then that triangle into a square.

PROBLEM XXIX. *To convert any given triangle into an isosceles triangle of equal area.*

SOLUTION. Let ABC be the given triangle, which is to be converted into an isosceles one.



1. Bisect the base AC in D, and from D draw the perpendicular DE.

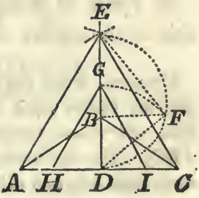
2. From the vertex, B, of the given triangle, draw BE, parallel to the base, AC.

3. From the point E, where this parallel meets the perpendicular, draw the straight lines EA, EC; EAC is the isosceles triangle sought.

DEMON. The triangles AEC and ABC are upon the same basis, AC, and between the same parallels (page 90, 3dly).

PROBLEM XXX. *To convert a given isosceles triangle into an equilateral one of equal area.* (This problem is intended for more advanced and elder pupils.)

SOLUTION. Let ABC be the given isosceles triangle.



1. Upon the base, AC, of the given triangle, draw the equilateral triangle AEC (problem I.); and through the vertices, E, B, of the two triangles, draw the straight

line EB, which evidently is perpendicular to AC, and bisects the last line in D (ABC, AEC, being isosceles triangles).

2. Upon ED describe the semicircle EFD, and from B draw the perpendicular BF, which meets the semicircle in F.

3. From D, with the radius DF, describe an arc, FG, cutting the line DE in G.

4. From G, draw the lines GH, GI, parallel to the sides of the equilateral triangle AEC; HGI is the equilateral triangle sought.

Demon. Since the line GH is parallel to AE, and GI parallel to EC, the angle GHI is equal to the angle EAI, and the angle GIH to the angle ECH (page 31). Thus the two triangles GHI, AEC, have two angles, GHI, GIH, in the one, equal to two angles, EAC, ECA, in the other, each to each; consequently they are similar to each other (page 73, 1st); and the triangle GHI must also be equilateral.

Suppose the lines DF and EF drawn; then DF is a mean proportional between DE and DB; for the triangle EDF is right-angled (see the Remark, page 155) in F, and if from the vertex of the right angle, the perpendicular FB is let fall upon the hypothenuse, the side DF is a mean proportional between the hypothenuse, ED, and the part, BD, of it, between the foot of the perpendicular, and the extremity, D, of the line FD (see page 75, 2dly); consequently we shall have the proportion

$$ED : DF = DF : BD;$$

and as DG is, by construction, made equal to DF,

$$ED : DG = DG : BD \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \text{(I.)}$$

Moreover, in the two similar triangles, ADE, HDG, the corresponding sides are proportional (page 70, 4thly); therefore we have the proportion

$$ED : DG = AD : HD \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \text{(II.)}$$

This last proportion has the first ratio common with the first proportion; consequently the two remaining ratios are in a geometrical proportion (Theory of Prop., Prin. 3d); that is, we have

$$AD : HD = DG : BD;$$

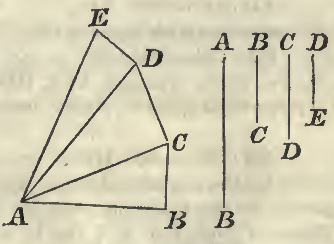
and as, in every geometrical proportion, the product of the means is

equal to that of the extremes (Theory of Prop., Principle 10th), we have HD multiplied by DG, equal to AD multiplied by BD; consequently, also, *half* the product of the line HD, multiplied by the line DG, equal to half the product of the line AD, multiplied by BD. But half the product of the line HD, multiplied by DG, is the area of the triangle HDG; because the triangle HDG is right-angled in D, therefore if HD is taken for the basis, DG is its height; and for the same reason is half the product of the line AD by BD, the area of the triangle ADB; consequently the areas of the two triangles, ADB and HDG, are equal to one another; and because the triangle HDG is equal to the triangle IDG, and the triangle ABD to the triangle CBD, the area of the whole triangle HIG is equal to the area of the whole triangle ABC; therefore the triangle HIG is the required equilateral triangle, which is equal, in area, to the given isosceles triangle, ABC.

Remark 1. If BD is greater than ED, then the perpendicular, BF, does not meet the semicircle. In this case, it is necessary to describe the semicircle on BD, and from E to draw the perpendicular. In this case, the points H, I, will not be situated in the line AC; but in its further extension.

Remark 2. From this and the preceding problems, it appears how any figure may be converted into an equilateral triangle; for it is only necessary first to convert the figure into a triangle, this triangle into an isosceles triangle, and the isosceles triangle into an equilateral one.

Problem XXXI. *To describe a square, which in area shall be equal to the sum of several given squares.*



Solution. Let AB, BC, CD, DE, be the sides of four squares; it is required to find a square which shall be equal to the sum of these four squares.

1. At the extremity, B, of the line AB, draw a perpendicular equal to BC, and join AC.

2. At the extremity, C, of the line AC, draw a perpendicular equal to CD, and join AD.

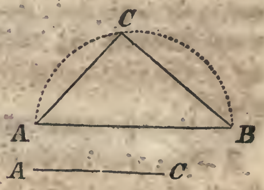
3. At the extremity, D, of the line AD, draw a perpendicular equal to DE, and join AE; the square upon AE is, in area, equal to the sum of the four squares upon the lines AB, BC, CD, DE.

Demon. The square upon the hypothenuse, AC, of the right-angled triangle ABC, is equal to the sum of the squares upon the two sides AB, BC (Query 6, Sect. III.); and for the same reason is the square upon AD equal to the sum of the squares upon CD and AC; consequently, also, to the squares upon CD, CB, and AB (the square upon AC being equal to the squares upon CB and AB); and finally the square upon AE is equal to the sum of the squares upon ED and AD; or, which is the same, to the sum of the squares upon DE, CD, CB, and AB

Problem XXXII. *To describe a square which shall be equal to the difference of two given squares.*

Solution. Let AB, AC be the sides of two squares.

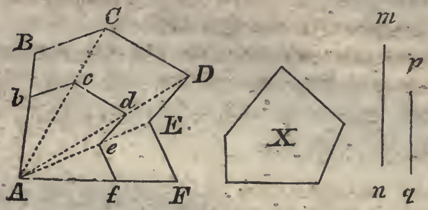
1. Upon the greater side, AB, as a diameter, describe a semicircle.



2. From A, within the semicircle, draw the line AC, equal to the given line AC, and join BC; then CB is the side of the square sought.

Demon. The triangle ABC is right-angled in C, and in every right-angled triangle, the square upon one of the sides, which include the right angle, is equal to the difference between the squares upon the hypothenuse and the other side.

Problem XXXIII. *To transform a given figure in such a way, that it may be similar to another figure.*



Solution. Let X be the given figure, and ABCDEF the one to which it is to be similar.

1. Convert the figure ABCDEF into a square (see the remark, page 166), and let its side be mn, so that the area of the square upon mn is equal to the area of the figure ABCDEF; convert, also, the figure X into a square, and let its side be pq, so that the area of the square upon pq shall be equal to the area of the figure X.

2. Take any side of the figure ABCDEF, say AF; and to the three lines, mn, pq, AF, find a fourth proportional (Problem X), which you cut off from AF. Let Af be this fourth proportional, so that we have the proportion

$$mn : pq = \mathrm{AF} : \mathrm{A}f.$$

3. Then draw the diagonals AE, AD, AC, and the lines fe, ed, dc, cb, parallel to the lines FE, ED, DC, CB; then A$bcdef$ will be the required figure, which in area is equal to the figure X, and is similar to the figure ABCDEF.

Demon. It is easily proved, that the figure A$bcdef$ is similar to ABCDEF. Further, we know that the areas of the two similar figures, ABCDEF, A$bcdef$, are to each other, as the areas of the squares upon the corresponding sides AF, Af, (see page 198); which may be expressed,

$$\mathrm{ABCDEF} : \mathrm{A}bcdef = \mathrm{AF} \times \mathrm{AF} : \mathrm{A}f \times \mathrm{A}f;$$

and as the sides AF and Af are (by construction 2) in proportion to the lines mn, pq, the squares upon these sides, and therefore the figures ABCDEF, $Abcdef$, themselves, are in proportion to the squares upon mn and pq; that is, we shall have the proportion

$$ABCDEF : Abcdef = mn \times mn : pq \times pq.$$

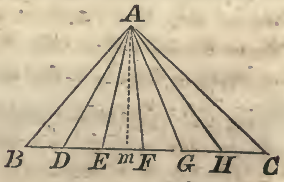
This proportion expresses, that the area of the figure ABCDEF is as many times greater than the area of the figure $Abcdef$, as the area of the square upon the line mn is greater than the area of the square upon the line pq; therefore, as the area of the figure ABCDEF is, by construction, equal to that of the square upon the line mn, the area of the figure $Abcdef$ is equal to that of the square upon the line pq. But the square upon pq is made equal to that of the figure X; therefore the area of the figure $Abcdef$ is also equal to that of the figure X; and the figure $Abcdef$ is the one required.

PART III.

Partition of figures by drawing.

Problem XXXIV. *To divide a triangle from one of the vertices into a given number of parts.*

Solution. Let ABC be the given triangle, which is to be divided, say, into six equal parts; let A be the vertex, from which the lines of division are to be drawn.



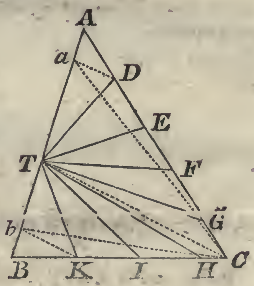
1. Divide the side BC, opposite the vertex A, into six equal parts, BD, DE, EF, FG, GH, HC.

2. From A to the points of division, D, E, F, G, H, draw the lines AD, AE, AF, AG, AH; the triangle ABC is divided into the six equal triangles, ABD, ADE, AEF, AFG, AGH, AHC.

DEMON. The triangles ABD, ADE, AEF, AFG, AGH, AHC, are, in area, equal to one another, because they have equal bases and the same height, Am (page 89).

Remark. If it is required to divide the triangle ABC according to a given proportion, it will only be necessary to divide the line BC in this proportion, and from A to draw lines to the points of division.

PROBLEM XXXV. *From a given point* in one of the sides *of a triangle, to divide it into a given number of equal parts.*



SOLUTION. Let ABC be the given triangle, which is to be divided into eight equal parts; the lines of division are to be drawn from T.

1. Make Aa and Bb equal to $\frac{1}{8}$ of AB, and from T draw the line TC to the vertex, C, of the triangle.

2. From a and b draw the lines aD, bK, parallel to TC, meeting the sides AC, BC, in D and K.

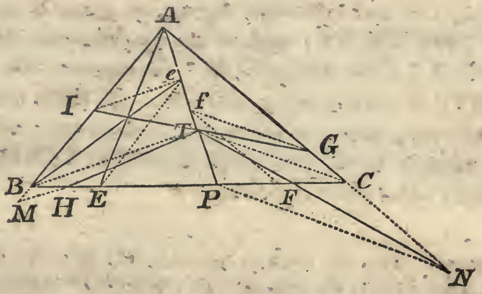
3. Upon AC, from A towards C, measure off the distance AD as many times as possible (in this case four times); and thus determine the points E, F, G; upon BC, in the direction from B towards C, also measure off the distance BK, as many times as is possible (here three times), and determine the points I, H.

4. From T draw the lines TD, TE, TF, TG, TH, TI, TK; then ATD, DTE, ETF, FTG, GTHC, HTI, ITK, KTB, are the eight equal parts of the triangle ABC.

Demon. Draw Ca; then the triangle AaC is $\frac{1}{8}$ of the triangle ABC; because, if AB is taken for the base of the triangle ABC, the base Aa of the triangle AaC is, by construction, $\frac{1}{8}$ of the base of the triangle ABC (see page 171). Now, the triangle aDC is equal to the triangle aDT; because these two triangles are upon the same base, aD, and between the same parallels, aD, TC; therefore (by adding to each of them the triangle aAD) the two triangles ADT and aAC are also equal; that is, ADT is also $\frac{1}{8}$ of the triangle ABC. In the same manner (by drawing the line bC) it may be proved that the triangle BKT is also $\frac{1}{8}$ of the triangle ABC. Further, the triangles ATD, DTE, ETF, FTG, are, by construction, all equal to one another, having equal bases and heights (see the demonstration to the last problem); and for the same reason are the triangles BTK, KTI, ITH equal to one another; therefore each of the seven triangles ATD, DTE, ETF, FTG, BTK, KTI, ITH, is $\frac{1}{8}$ of the triangle ABC; consequently the quadrilateral GTHC must be the remaining one eighth of the triangle ABC; and the area of the triangle ABC is divided into eight equal parts.

Problem XXXVI. *To divide a triangle, from a given point within it, into a given number of equal parts.*

[This problem is intended for elder pupils.]



Solution. Let ABC be the given triangle, which is to be divided, say, into five equal parts; T the point from which the lines of division are to be drawn.

1. Through the point T and the vertex A of the triangle, draw the line AT.

2. Take any side of the triangle, say BC, and make, when, as here, the triangle is to be divided into five equal parts, BE and CF equal to $\frac{1}{5}$ of BC, and draw the lines Ee, Ff, parallel to the sides AB, AC; these lines will meet the line AT in the points e and f.

3. From T draw the lines TB, TC, to the vertices B and C of the triangle ABC, and from e and f, the lines eI, fG parallel to TB, TC.

4. Join TI, TG; then each of the triangles ATI, ATG, is $\frac{1}{5}$ of the given triangle ABC.

5. In order to determine the other points of division, it is only necessary to cut off from the sides AB, AC, as many distances, equal to AI, AG, respectively, as is possible (see the solution of the last problem), and in the case where this can no longer be effected, or in which, as in the figure, this is impossible, proceed in the following manner:

a. Extend the two sides AB, AC, and then make IM equal to AI, and GN equal to AG.

b. From M and N draw the lines, MH, NP, parallel to BT and CT, and determine thereby the points H and P.

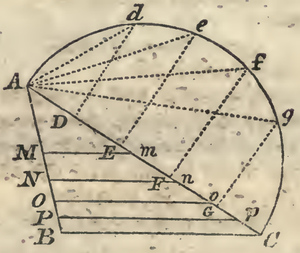
6. Draw TH, TP; each of the quadrilaterals IBHT, GCPT, is $\frac{1}{5}$ of the triangle ABC; consequently the triangle HTP is the remaining fifth of it. (If HTB were not the last part, then it would merely be necessary to divide this triangle by the rule given in problem XXXIV, into as many equal parts as necessary.)

DEMON. Draw the auxiliary lines AE, Be; then the triangle ABE is one fifth of the triangle ABC; because BE is one fifth of the basis BC (problem XXXIV); further, the triangle ABE is equal to the triangle ABe; because these two triangles are upon the same basis, AB, and, by construction 2, between the same parallels, AB, Ee; and the last triangle, ABe, is also equal to the triangle ATI; because the triangle ABe consists of the two triangles AIe and IeB, which are equal to the two triangles AIe and ITe (the

two triangles ITe and IeB being upon the same base, Ie, and, by construction 3, between the same parallels, Ie, BT); therefore the triangle AIT is also one fifth of the triangle ABC; and in the same manner it can be proved that ATG is one fifth of the triangle ABC.

Further, the triangle GNT is equal to the triangle AGT (the basis GN being made equal to the basis AG, and the vertical point T being common to both triangles); and the triangle GNT is equal to the quadrilateral CGTP; because the triangle CTN is equal to the triangle CTP (these two triangles being, by construction, upon the same base, TC, and between the same parallels, TC, PN); therefore the area of the quadrilateral CGTP is also one fifth of the triangle ABC; and in the same manner it may be proved that the area of the quadrilateral IBHT is one fifth of the triangle ABC; and as the two triangles AGT, AIT, together with the two quadrilaterals CGTP, IBHT, make four fifths of the triangle ABC, the triangle HPT must be the remaining one fifth of it.

Problem XXXVII. *To divide a given triangle into a given number of equal parts, and in such a way, that the lines of division shall be parallel to a given side of the triangle.*



Solution. Let ABC be the given triangle; let the number of the parts, into which it is required to be divided, be five, and BC the side to which the lines of division are to be parallel.

1. Upon one of the other two sides, say AC, describe a semicircle, and divide the side AC into as many equal

parts as the triangle is to be divided into; consequently, in the present case, into five; the points of division are D, E, F, G.

2. From these points of division draw the perpendiculars D*d*, E*e*, F*f*, G*g*, meeting the semicircle in the points *d*, *e*, *f*, *g*.

3. From A draw A*d*, A*e*, A*f*, A*g*; then make A*m* equal to A*d*, A*n* equal to A*e*, and so on, and by these means determine the points *m*, *n*, *o*, *p*.

4. From these points draw the lines *m*M, *n*N, *o*O, *p*P, parallel to the side BC; then AM*m*, M*m*N*n*, N*n*O*o*, O*o*P*p*, P*p*BC are the five equal parts of the triangle ABC, which were sought.

DEMON. Imagine the line *d*C drawn; the triangle A*d*C, inscribed in the semicircle, is right-angled in *d*; consequently we have the proportion

$$AD : Ad = Ad : AC;$$

and as, in every geometrical proportion, the product of the mean terms is equal to that of the extremes,

$$Ad \times Ad = AD \times AC;$$

consequently, also,

$$Am \times Am = AD \times AC$$

(because A*m* is made equal to A*d*).

Further, the triangles AM*m*, ABC, are similar, because the line M*m* is drawn parallel to the side BC in the triangle ABC (Query 16, Sect. II.); and as the areas of similar triangles are to each other as the areas of the squares upon the corresponding sides (Query 8, page 97), we have the proportion

$$\text{triangle } ABC : \text{triangle } AMm = AC \times AC : Am \times Am;$$

therefore, also,

$$\text{triangle } ABC : \text{triangle } AMm = AC \times AC : AC \times AD$$

(because A*m* × A*m* is equal to AC × AD).

The last proportion expresses, that the area of the triangle ABC is as many times greater than the area of the triangle AM*m*, as AC times the side AC itself is greater than AC times the side AD; or, which is the same, as AC is greater than AD (Prin. 7th of Geom.

Prop. page 63). But the side AD is, by construction, one fifth of AC; therefore the area of the triangle AM*m* is also one fifth of the area of the triangle ABC. In like manner it may be proved, that the triangle AN*n* is two fifths of the triangle ABC; the triangle AO*o* three fifths, and the triangle AP*p* four fifths of it, from which the rest follows of course.

Remark. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will merely be necessary, as may be readily seen from the above, to divide the line AC according to this proportion, and then proceed as has been already shown.

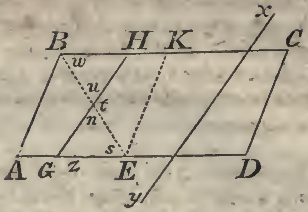
PROBLEM XXXVIII. *To divide a parallelogram into a given number of equal parts, and in such a way, that the lines of division may be parallel to two opposite sides of the parallelogram.*

SOLUTION. Let ABCD be the given parallelogram; let the number of parts be six; and let AB, CD, be the sides, to which the lines of division shall be parallel.

Divide one of the two other sides, say AD, into six equal parts, in E, F, G, H, I, and from these points draw the lines E*e*, F*f*, G*g*, H*h*, I*i*, parallel to the sides AB, CD; then the division is done.

Remark. If it is required to divide the parallelogram according to a given proportion, it will merely be necessary, instead of dividing the line AD into equal parts, to divide it according to the given proportion, and then proceed as before.

PROBLEM XXXIX. *To divide a parallelogram, according to a given proportion, by a line which shall be parallel to a line given in position.*



SOLUTION. Let ABCD be the parallelogram to be divided.

1. Divide one of its sides, say AD, according to the given proportion; let the point of division be in z.

2. Make zE equal to the distance Az, and draw BE. Now, if the line BE has the required position, the triangle ABE and the quadrilateral BCDE are the parts sought.

3. But if the line of division is required to be parallel to the line xy, bisect the line BE in t, and through this point draw the line GH parallel to xy; then the two quadrilaterals ABHG, HCDG, will be the required parts.

DEMON. Draw EK parallel to AB. Then the two parallelograms ABEK, ABCD, having the same height, their areas are in proportion to their bases, AE, AD (see page 88, 7th); that is, we have

parallel. ABEK : parallel. ABCD = AE : AD;

therefore

$\frac{1}{2}$ of parallel. ABEK : parallel. ABCD = $\frac{1}{2}$ of AE : AD;

and because the triangle AEB is equal to half the parallelogram ABEK, and half of AE is, by construction, equal to Az, we have

triangle AEB : parallel. ABCD = Az : AD.

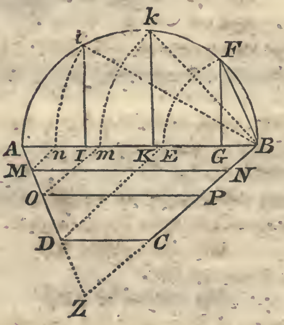
This last proportion expresses that the area of the parallelogram ABCD is as many times greater than the area of the triangle ABE, as the line AD is greater than Az; consequently if BE has the required position, the triangle ABE is one of the required parts, and therefore the trapezoid BEDC the other.

Further, the line BE is (by construction 3) bisected; the an-

gles u and n are opposite angles at the vertex, and w and s are alternate angles (page 31, 2d); therefore the triangle BtH, having the side Bt, and the two adjacent angles, w and u, equal to the side tE, and the two adjacent angles, n and s, in the triangle GtE, these two triangles are equal to one another; consequently the area of the trapezoid ABGH (composed of the quadrilateral ABGt, and the triangle BtH) is equal to the area of the triangle ABE (composed of the same quadrilateral ABGt and the equal triangle GtE), which proves the correctness of construction 3.

Problem XL. *To divide a trapezoid into a given number of equal parts, so that the lines of division may be parallel to the parallel sides of that trapezoid.*

[This problem may be omitted by the younger pupils.]



Solution. Let ABCD be the given trapezoid which is to be divided into three equal parts.

1. Upon AB, the greater of the two parallel sides, describe a semicircle; draw DE parallel to CB; and from B, with the radius BE, describe the arc of a circle, EF, cutting the semicircle in F.

2. From F draw FG perpendicular to AB, and divide

the part AG of the line AB, into three equal parts in K and I; from these points draw the perpendiculars Kk, Ii.

3. Upon AB, from B towards A, take the distances Bm, Bn, equal to Bk, Bi; from the points m and n, draw the lines mO, nM, parallel to BC; and from the points O, M, in which these parallels meet the side AD, the lines MN, OP, parallel to AB; then ABNM, MNPO, OPCD, are the three required parts of the trapezoid ABCD

DEMON. Extend the lines AD, BC, until they meet in Z. Then the triangles DCZ, OPZ, MNZ, ABZ, are all similar to each other (page 70); further, we have (by construction 3)

DC equal to BE and to BF,
OP " " Bm " " Bk,
MN " " Bn " " Bi.

The areas of the two similar triangles OPZ, CDZ, are in the ratio of the squares upon the corresponding sides; that is, we have the proportion

$$\text{triangle OPZ} : \text{triangle DCZ} = OP \times OP : CD \times CD;$$

and since OP is equal to Bk, and CD to BF, also

$$\text{triangle OPZ} : \text{triangle DCZ} = Bk \times Bk : BF \times BF.$$

Imagine AF and FB joined; the triangle AFB would be right-angled in F, and we should have the proportion

$$BG : BF = BF : AB;$$

and for the same reason we have

$$BK : Bk = Bk : AB.$$

Taking the product of the mean and extreme terms of the two last proportions, we have

$BG \times AB$ equal to $BF \times BF$, and
$BK \times AB$ " " $Bk \times Bk$

Let us now take our first proportion,

$$\text{triangle OPZ} : \text{triangle DCZ} = Bk \times Bk : BF \times BF;$$

and let us write $BG \times AB$, instead of $BF \times BF$ (its equal), and $BK \times AB$, instead of $Bk \times Bk$, and we shall have

$$\text{triangle OPZ} : \text{triangle DCZ} = AB \times BK : AB \times BG,$$

whence

$$\text{triangle OPZ} : \text{triangle DCZ} = BK : BG;$$

consequently, also,

triangle OPZ — triangle DCZ : triangle DCZ = BK — BG : GB; (Principle 6th of Geom. Prop. page 62); which is read thus:

triangle OPZ, less the triangle DCZ, is to the triangle DCZ, as the line BK, less the line BG, is to the line BG;

that is, trapezoid DOPC : triangle DCZ = GK : BG;

and as GK is (by construction 2) equal to $\frac{1}{3}$ of AG,

trapezoid DOPC : triangle DCZ = $\frac{1}{3}$ AG : BG.

In like manner it may be proved that

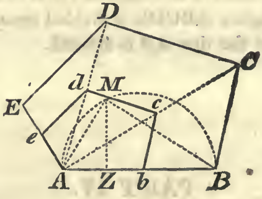
trapezoid DMNC : triangle DCZ = $\frac{2}{3}$ AG : BG and

trapezoid DABC : triangle DCZ = AG : BG.

These proportions express that the three trapezoids DOPC, DMNC, DABC, are to each other in the same proportion as one third is to two thirds to three thirds; or, which is the same, as one is to two, to three; whence the rest of the demonstration follows of course.

Remark. If it is required to divide the trapezoid ABCD not into equal parts, but according to a given proportion, it will only be necessary to divide the line AG in this proportion, and then proceed as before.

Problem XLI. *To divide a given figure into two parts according to a given proportion, and in such a way, that one of the parts may be similar to the whole figure.*



Solution. Let ABCDE be the given figure.

1. Divide one side of the figure, say AB, according to the given proportion; let the point of division be Z.

2. Upon AB, as a diameter, describe a semicircle, and

16

from Z draw the perpendicular ZM, meeting the semicircle in M.

3 Make $Ab = AM$, and upon Ab describe a figure, $Abcde$, which is similar to the given one, ABCDE (see Problem XXXIII); the line $bcde$ divides the figure in the manner required.

DEMON. The areas of the two similar figures $Abcde$, ABCDE, are to each other, as the squares upon their corresponding sides (page 98); therefore we have the proportion

$$ABCDE : Abcde = AB \times AB : Ab \times Ab.$$

Draw AM and BM; then AM is a mean proportional between AZ and AB; that is, we have

$$AZ : AM = AM : AB;$$

and as Ab is, by construction, equal to AM,

$$AZ : Ab = Ab : AB;$$

consequently the product $Ab \times Ab$ is equal to $AZ \times AB$.

Writing $AZ \times AB$, instead of $Ab \times Ab$ (its equal), in the first proportion, we have

$$ABCDE : Abcde = AB \times AB : AB \times AZ.$$

Hence $ABCDE : Abcde = AB : AZ$; and therefore

$$ABCDE - Abcde : Abcde = AB - AZ : AZ;$$

which is read thus:

ABCDE, less $Abcde$, is to $Abcde$ as AB, less AZ, is to AZ; that is,

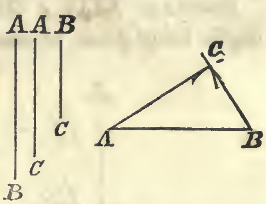
BCDE$edcb$ is to $Abcde$ as ZB is to AZ;

consequently the figure ABCDE is divided according to the given proportion in which the line AB is divided.

PART IV.

Construction of triangles.

PROBLEM XLII. *The three sides of a triangle being given, to construct the triangle.*



SOLUTION. Let AB, AC, BC, be the three given sides of the triangle.

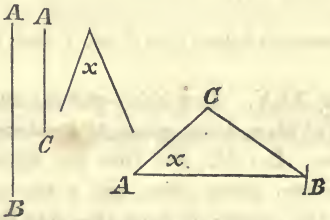
1. Take any side, say AB, and from A as a centre, with the radius AC, describe an arc of a circle.

2. From B, as a centre, with the radius BC, describe another arc, cutting the first.

4. From the point of intersection C, draw the straight lines CA, CB; the triangle ABC is the one required.

The demonstration follows immediately from Query 4th, Sect. II.

PROBLEM XLIII. *Two sides, and the angle included by them, being given, to construct the triangle.*



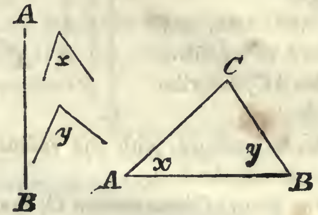
SOLUTION. Let AB, AC, be the two given sides, and x the angle included by them.

1. Construct an angle equal to the angle x (Problem VI); make one of the legs equal to the side AB, and the other to the side AC.

2. Join BC; the triangle ABC is the one required.

The demonstration follows from Query 1, Sect. II.

Problem XLIV. *One side and the two adjacent angles being given, to construct the triangle*



Solution. Let AB be the given side, and x and y the two adjacent angles.

1. At the two extremities of the line AB, construct the angles x and y, and extend their legs, AC, BC, until they meet in the point C; the triangle ABC is the one required.

The demonstration follows from Query 2, Sect. II.

Problem XLV. *Two sides, and the angle opposite to the greater of them, being given, to construct the triangle.*

Solution. Let AC, BC (see the figure to Problem XLIII) be two given sides, and x the angle, which is opposite to the greater of them (the side BC).

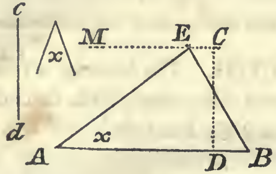
1. Upon an indefinite straight line construct an angle equal to the angle x.

2. Make the leg AC of this angle equal to the smaller side AC, and from C as a centre, with the radius CB equal to the greater side, describe an arc of a circle, cutting the line AB in the point B.

3. Join BC; the triangle ABC is the one required.

The demonstration follows from Query 10th, Sect. II.

PROBLEM XLVI. *The basis of a triangle, one of the adjacent angles, and the height being given, to construct the triangle.*



SOLUTION. Let AB be the given basis, x one of the adjacent angles, and *cd* the height.

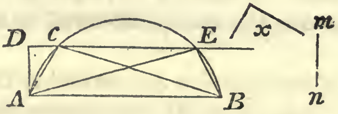
1. In any point of the line AB, draw a perpendicular, CD, equal to *cd;* and through C a line parallel to AB.

2. In A make an angle equal to the given angle x, and extend the leg AE until it meets the line MC.

3. Join EB; the triangle AEB is the one required.

The demonstration is sufficiently evident from the construction.

PROBLEM XLVII. *The basis, the angle opposite to it, and the height of a triangle being given, to construct the triangle.*



SOLUTION. Let AB be the given base, x the angle opposite to it, and *mn* the height of the triangle.

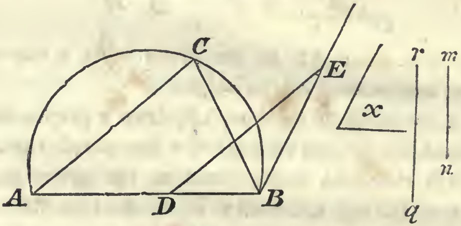
1. Upon the base AB describe a segment of a circle containing a given angle x (see Problem XVII).

2. In A draw a perpendicular, AD, equal to the given height *mn*, and through D draw DE parallel to AB.

3. From C and E, where this parallel cuts the segment, draw the straight lines CA, CB, EA, BE; either of the two triangles ACB, AEB, will be the one required.

The demonstration follows from the construction.

PROBLEM XLVIII. *The basis of a triangle, the angle opposite to it, and the ratio of the two other sides being given, to construct the triangle.*



SOLUTION. Let AB be the given basis, x the angle opposite to it; and let the two remaining sides bear to each other the same ratio which exists between the two lines mn and rq.

1. Upon AB describe a segment of a circle capable of the given angle x (see Problem XVII).

2. In B make an angle, ABE, equal to the angle x; make BE equal to the line rq, BD equal to mn, and join DE.

3. From A draw the line AC parallel to DE, and from the point C, where it meets the segment, draw the line CB; the triangle ABC is the one required.

DEMON. The triangle ABC is similar to the triangle DBE; because the two angles CAB and ACB, in the one, are equal to the two angles BDE, DBE, in the other, each to each* (page 73, 1st); therefore we have the proportion

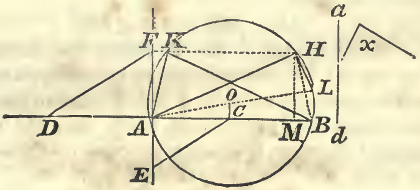
* CAB and EDB being alternate angles, and each of the angles, ACB, DBE, being made equal to the given angle x.

$$AC : BC = BE : BD,$$

which expresses that the two sides, AC, BC, of the triangle are in the same ratio as the sides BE, BD, of the triangle DEB; consequently they are also as the lines *rq*, *mn*; because BE and BD are, by construction, equal to *mn*, *rq*. The rest of the demonstration is evident from the construction.

PROBLEM XLIX. *The basis of a triangle, the angle opposite to it, and the square, which, in area, is equal to the rectangle of the two remaining sides, being given, to construct the triangle.**

[Let the younger pupils omit this problem.]



SOLUTION. Let AB be the given base, x the angle opposite to it, and *ad* the side of the square, equal to the rectangle of the two remaining sides.

1. Upon AB construct the segment, AKHB, of a circle, capable of the given angle x.

2. Extend AB towards D, and in A draw the perpendicular AF.

3. Make AC equal to the radius AO, AD to the side *ad* of the given square, and AE to half of AD; join EC, and from D draw DF parallel to EC.

4. Through the point F, where this parallel meets the perpendicular, draw FH parallel to AB; and from the points K and H, where this meets the segment, the lines

* By the rectangle of the two remaining sides is meant a rectangle, whose base is one of these sides, and whose height is the other.

AK, KB, AH, HB; then either of the two triangles AKB, AHB, is the one required.

Demon. Draw the diameter AL, and from either of the points K, H, say H, let fall the perpendicular HM upon AB. The triangle ALH is similar to the triangle MBH; for the triangle ALH being inscribed in a semicircle, each of these triangles is right-angled, and the two angles ALH, ABH, are equal; because both of them measure half as many degrees as the arc AKH (page 111, 1st); therefore the remaining angles, HAL and MHB, are also equal (page 73, 1st); and the corresponding sides of the two triangles ALH, MBH, are in the geometrical proportion

$$AH : AL = HM : HB;$$

consequently we have

$$AL \times HM = AH \times HB.$$

This proportion expresses, that the area of the rectangle, which has for its base the diameter AL, and its height equal to the height HM of the right-angled triangle AHB, is equal to the area of the rectangle, which has the side AH for its base, and the side HB for its height.* Further, it is easy to perceive that, from the similar triangles ACE, ADF, we have the proportion

$$AD : AF = AC : AE;$$

consequently, also,

$$AD : AF = 2AC : 2AE,$$

therefore,

$$2AC \times AF = 2AE \times AD; \text{ or}$$

$$\text{diam. } AL \times AF = ad \times ad;$$

(because AC is equal to the radius Ao of the circle, and AE is half of AD, and AD is equal to ad).

From this proportion it follows, that the area of the square upon ad, is equal to that of the rectangle of AL by AF, or MH its equal (see the figure); and as the rectangle AH by HB is equal to that of AL by HM, as we have proved above, it must also be equal to the square upon ad. The same may be proved of the rectangle of the two sides AK, KB, of the triangle AKB.

The rest of the demonstration is sufficiently evident from the construction.

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* For the area of a rectangle is found by multiplying the base by the height.

APPENDIX.

Containing Exercises for the Slate.

1. The side of a square being 12 feet, what is its area?

2. What, if the side is 12 rods, miles, &c.?

3. What is the side of a square, whose area is one square foot?

4. What, that of a square, whose area is one square yard, rod, mile, &c.?

5. What, that of a square of 4, 9, 16, 25, 36, 49, 64, 81, 100 sqare feet?

6. What is the area of a rectangle, whose base is 50 feet 3 inches, and whose height 10 feet 4 inches?

7. What, that of a rectangle, whose base is 40 feet 3 inches, and whose height is $12\frac{1}{2}$ feet?

8. If the area of a rectangle is 240 square feet 19 square inches, and its basis measures 30 feet, what is its height?

9. What is the basis of a rectangle, whose height is 10 feet, and whose area is 40 square feet?

10. What is the area of a rectangle, whose basis is 4 feet, and whose height is 3 inches?

11. What is the area of a parallelogram of 10 feet basis, and 3 feet 4 inches high?

12. The height of a parallelogram is 5 feet, and the area 40 square feet: what is its basis?

13. The sum of the two parallel sides of a trapezoid is 12 feet, and their distance 3 feet 4 inches: what is the area of the trapezoid?

14. The area of a trapezoid is 24 square feet, and its height is 4 inches, 3 seconds: what is the sum of its bases?

15. What is the difference between a triangle whose basis is 10 feet 3 inches, and height 9 feet, and a triangle of 3 feet basis, and 11 inches height?

16. What is the difference between a trapezoid, the sum of the two parallel sides of which is 14 feet 3 inches, and height 9 inches, and a square upon 9 inches?

17. What is the sum of the areas of a triangle of 3 feet basis, and 9 inches height; a square upon 14 feet 3 inches, and a rectangle whose basis is 3 feet 2 inches, and height 1 foot 4 inches?

18. What is the area of a circle, whose radius is 9 inches?

19. What that of a circle, whose radius is 10 feet?

20. What that of a circle, whose radius is 9 feet 6 inches?

21. The area of a circle is 240 square feet: what is its radius or diameter?*

22. The radius of a circle is 5 feet 8 inches: what is its circumference?

23. What is the length of an arc of 14 degrees 29 minutes 24 seconds, in a circle whose radius is 14 inches?

24. What that of an arc of 6 degrees 9 seconds, in a circle whose radius is 1 foot?

25. What that of an arc of 9 seconds, in a circle whose radius is 1 mile?

26. What is the area of a sector of 15 degrees, in a circle whose radius is 3 feet?

27. What that of a sector of 19 degrees 45 minutes, in a circle whose radius is 1 foot 3 inches?

The teacher may now vary and multiply these questions.

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* Divide the area by π (see the note to page 132), and extract the square root of the quotient, the answer is the radius of the circle.

END.